

A THEORETICAL AND EXPERIMENTAL STUDY
OF THE THREE JET INTERACTION

Gerardo Hiriart Le-Bert

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Gerardo Hiriart Le-Bert

April 1970

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A Theoretical and Experimental Study
of the Three Jet Interaction

by

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Submitted in partial fulfillment of the
requirements for the degrees of

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ABSTRACT

The interaction of three jets is studied both theoretically and experimentally. The angle of deflection and the velocity and turbulence distributions were determined through the use of a hot-wire anemometer. The free-streamline theory has been applied to the determination of the jet deflection and the results so obtained were found to be in good agreement with those obtained experimentally. The study is a convincing evidence of the fact that a proper combination of inviscid flow analysis together with experimental results can provide the necessary guidance for the design of proportional amplifiers.

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LIST OF SYMBOLS

a	Power nozzle width
b	Control nozzles width
CF(P)	Pressure correction factor for rotameter
d	Resultant jet's width at vena contracta
\mathcal{J}	Imaginary part of the expression
i	$\sqrt{-1}$
PA	Power jet, stagnation pressure
PC,PG	Control jets, stagnation pressure
q	Magnitude of local velocity
\mathcal{R}	Real part of the expression
r	Distance from pivoting point
Re	Reynolds number
s	Setback of control nozzles
t	Coordinate in t-plane
U	Instantaneous velocity
U	Local mean velocity
$\sqrt{u^2}$	R.M.S. value of the fluctuating component of velocity
VA	Power jet velocity
VC	Left control jet velocity
VG	Right control jet velocity
VRO	Reference velocity $VRO = 0.3 VA$
VRI	Reference velocity $VRI = 0.4 VA$
Vj	Resultant jet velocity

β	Complement of the deflection angle
δ	Deflection angle of resultant jet
Ω	Coordinate in the Ω -plane

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I. INTRODUCTION

The interaction of compressible or incompressible jets has been subject to study for at least the past fifty years for the purpose of predicting the velocity and turbulence distributions in the combined jets and in general for the purpose of predicting the performance characteristics of all sorts of machinery where such jet interaction plays a significant role. Most of the theoretical analyses have been restricted to the study of the interaction of jets of an inviscid fluid, and the complex effects of viscosity and the effects of the bounding walls have been largely ignored.

The advance of the fluidics technology and the discovery, among others, of proportional amplifiers have renewed a strong interest in both the theoretical and experimental investigation of jet interaction. Proportional amplifiers were designed and operated either through the use of the "pressure control" concept or through the use of "momentum control" concept. In either case, both the single-sided and double-sided devices were considered. The pressure controlled device is operated on the basis of the differential pressure between the two cavities placed on the two sides of the resultant jet. The vents are not open to the atmosphere. The momentum controlled proportional amplifier operates, as its name implies, on the momentum balance of the combining jets. Thus, it is apparent that, whereas the pressure-controlled amplifier depends primarily on pressure to deflect the jet, the momentum-controlled amplifier depends primarily on momentum. For certain configurations where the venting of the side cavities is not complete or where the entrainment from the two sides of the jet is not completely

attainable, then it is necessary to take both pressure and momentum effects into account in attempting to design amplifiers optimized for particular conditions. It should, in passing, be noted that the use of both control mechanisms tends to produce devices whose band-widths are relatively small, even though they may have significantly higher pressure gains. We shall confine our attention here to a thorough study of the momentum-controlled devices or beam-deflection amplifiers, both for the purpose of understanding their fluid mechanics and for the purpose of contributing to a predictable design of proportional amplifiers.

The literature concerning the fundamental behavior of beam-deflection devices is still relatively limited but is growing rapidly with the increasing realization of the design limitations of current fluidics technology.

The prediction of the performance of such devices depends on a knowledge of the exact mode of interaction of the jets. There are, obviously, many possible geometric and kinematic variables for the interaction process and it is not economically feasible to experimentally investigate the characteristics of the mode of operation of each and every possible combination. This realization necessarily led, in the first decade of fluidics, to the evolution of several flow models with varying degrees of approximations and complexities. There are, to be sure, essential differences, as well as similarities, among these models, and all must be carefully assessed before a design choice is made.

Dexter [Ref. 1] employed the "submerged jet" model, assuming that viscous mixing with the surrounding fluid has reached the stage where the velocity profile of the deflected jet is of Gaussian form. Moynihan and Reilly [Ref. 2] employed the "self preservation" model in which each jet is assumed to have retained its identity during interaction and is

not subjected to any viscous effects as it proceeds downstream. Douglas and Neve [Ref. 3] extended the works in the references just cited by applying the two models to the appropriate regions of the resultant jet.

The present work presents the theoretical results obtained through the use of the free-streamline theory for symmetric control-jet configurations and the experimental results obtained with a hot-wire anemometer.

II. THEORETICAL CONSIDERATIONS

As in all applications of the free-streamline theory, the pressure and magnitude of velocity along the free-stream lines are assumed constant. It is further assumed that both the pressure and velocity are continuous along the separation lines between the power jet and the control jets.

Experiments show that each control jet retains a large measure of its integrity, and the power jet undergoes an acceleration or a "vena contracta" effect during the initial face of the interaction. The combined jet attains a Gaussian profile only after a distance of two to six nozzle widths from the pivoting point, depending on the flow ratio and the geometry of the system.

It is apparent from the foregoing that the flow in the region in which the momentum-exchange and the turbulent mixing take place does not, strictly speaking, follow the assumptions made in the free-streamline theory. The complicated interaction between the jets, and the pressure of turbulent mixing, make the analytical predictions of three-dimensional flow field extremely difficult. To obtain analytical solutions, a simplified incompressible flow model, which retains some of the major features of the actual flow field, must be employed. Although the probable error introduced through the use of such a model is very difficult to estimate, previous analyses place a reasonable degree of confidence in the applicability of the model to certain features of the jet interaction phenomenon.

A. TRANSFORMATION PLANES

A schematic drawing of the flow configuration, together with the

resultant jet, is shown in Fig. 1a. This configuration may be transformed to an Ω -plane through the use of the Planck's transformation with the usual assumptions of the free-streamline theory. Thus, writing

$$\Omega = \ln \frac{v_j}{q} + i\theta \quad (1)$$

and carrying out the necessary manipulations, one obtains Fig. 1b.

It is a well-known fact that any polygon, such as the one shown in Fig. 1b, may be transformed to the upper or the lower half of a real plane through the use of Schwartz-Christoffel transformation given by

$$\Omega = \int \frac{M dt}{(t-z_1)^{\alpha_1/\pi} (t-z_2)^{\alpha_2/\pi} (t-z_3)^{\alpha_3/\pi} (t-z_4)^{\alpha_4/\pi} (t-z_5)^{\alpha_5/\pi}} + N \quad (2)$$

where M and N are constants to be determined and z_i and α_i are given by

$$z_1=0 \quad z_2=1 \quad z_3=-1 \quad z_4=k \quad z_5=-k$$

$$\alpha_1=-\pi \quad \alpha_2=\frac{\pi}{2} \quad \alpha_3=\frac{\pi}{2} \quad \alpha_4=\pi \quad \alpha_5=\pi$$

Then one has

$$\Omega = \int \frac{M t dt}{(t^2-k^2) \sqrt{t^2-1}} + N \quad (3)$$

The evaluation of this integral is shown in detail in Appendix I.

The result is that

$$\Omega = \ln \left(\frac{\sqrt{1-k^2} + \sqrt{1-t^2}}{\sqrt{t^2-k^2}} \right) \quad \text{for } t \leq -k \quad (4)$$

$$\Omega = \ln \left(\frac{\sqrt{1-k^2} + \sqrt{1-t^2}}{-\sqrt{t^2-k^2}} \right) \quad \text{for } -k \leq t \leq 1 \quad (5)$$

$$\Omega = \ln \left(\frac{-\sqrt{1-k^2} + \sqrt{1-t^2}}{\sqrt{t^2-k^2}} \right) \quad \text{for } t \geq 1 \quad (6)$$

Table I gives the corresponding values between the points in the Ω -plane and the t-plane shown in Fig. 1c.

The relationships between the parameters shown in the t-plane and various velocity ratios are obtained as follows:

Through the use of the conditions at point A

$$-\ln \frac{v_A}{v_j} + i \frac{\pi}{2} = \ln \left(\frac{\sqrt{1-k^2} + 1}{-\sqrt{1-k^2}} \right)$$

one has

$$\eta = \frac{v_A}{v_j} = \frac{k}{1 + \sqrt{1-k^2}} \quad (7)$$

Through the use of the conditions at point C

$$-\ln \frac{v_C}{v_j} + i0 = \ln \left(\frac{\sqrt{1-k^2} + \sqrt{1-c^2}}{\sqrt{c^2-k^2}} \right)$$

one has

$$\alpha = \frac{v_C}{v_j} = \frac{\sqrt{c^2-k^2}}{\sqrt{1-k^2} + \sqrt{1-c^2}} \quad (8)$$

Through the use of the conditions at point G

$$-\ln \frac{v_G}{v_j} + i\pi = \ln \left(\frac{\sqrt{1-k^2} + \sqrt{1-g^2}}{-\sqrt{g^2-k^2}} \right)$$

one has

$$\gamma = \frac{v_G}{v_j} = \frac{\sqrt{g^2-k^2}}{\sqrt{1-k^2} + \sqrt{1-g^2}} \quad (9)$$

Through the use of the conditions at point E

$$i\beta = \ln \left(\frac{\sqrt{1-k^2} + \sqrt{1-f^2}}{\sqrt{f^2-k^2}} \right)$$

one has

$$\sin \beta = \sqrt{\frac{f^2-1}{f^2-k^2}} \quad \text{and} \quad \cos \beta = \sqrt{\frac{1-k^2}{f^2-k^2}} \quad (10)$$

Finally, through the use of the conditions on the free streamline, we obtain

$$\Omega = i\theta = \ln \left(\frac{\pm \sqrt{1-k^2} + \sqrt{1-t^2}}{\sqrt{t^2-k^2}} \right)$$

or

$$e^{i\theta} = \pm \sqrt{\frac{1-k^2}{t^2-k^2}} + i \sqrt{\frac{t^2-1}{t^2-k^2}}$$

$$e^{\Omega} = \cos \theta + i \sin \theta \quad (11)$$

It should be noted that the conservation of mass yields

$$d = (\alpha + \gamma) b + \eta a$$

The t -plane is comprised of three sources and one sink. Thus, the complex potential for the flow in the upper half of the t -plane may be written in terms of the known potentials of sources and sinks as:

$$w = \frac{V_j}{\pi} \ln(t+f) - \frac{VA}{\pi} \ln t - \frac{VC}{\pi} \ln(t+c) - \frac{VG}{\pi} \ln(t-g) \quad (12)$$

where $w = \phi + i\psi$ and ϕ is the complex potential and ψ the stream function.

Taking the derivative of w with respect to t , making use of Eqs. (7), (8), and (9), and the conservation of mass relationship, one has

$$\frac{dw}{dt} = - \frac{V_j}{\pi} \left[- \frac{d}{t+f} + \frac{\eta a}{t} + \frac{\alpha b}{t+c} + \frac{\gamma b}{t-g} \right] \quad (13)$$

The distances between various points in the z -plane, the contraction coefficient, and the angle of deflection may be evaluated as follows:

$$\overline{DF} = \mathcal{R} \left[\int dz \right] \quad (14)$$

and noting that $\overline{DF} = a + 2s$, one has

$$a + 2s = \mathcal{R} \left[\int_{-1}^{+1} e^{\tau} \left(-\frac{1}{v_j} \frac{dw}{dt} \right) dt \right]$$

Replacing $\frac{dw}{dt}$ from Eq. (13) and Ω from Eqs. (4) and (6), and taking the Cauchy principal value of the integral to account for the discontinuity at $t = -f$, we finally have,

$$\begin{aligned} a + 2s = d \sin \beta + \lim_{\varepsilon \rightarrow 0} \left[\frac{\sqrt{1-k^2}}{\pi} \int_{-1}^{-f+\varepsilon} \left(\eta \frac{a}{t} + \frac{\alpha b}{t+c} + \frac{\gamma b}{t-g} - \frac{d}{t+f} \right) \frac{dt}{\sqrt{t^2-1}} \right] \\ + \lim_{\substack{\varepsilon \rightarrow 0 \\ Q \rightarrow \infty}} \left[\frac{\sqrt{1-k^2}}{\pi} \int_{-f-\varepsilon}^{-Q} \left(\eta \frac{a}{t} + \frac{\alpha b}{t+c} + \frac{\gamma b}{t-g} - \frac{d}{t+f} \right) \frac{dt}{\sqrt{t^2-1}} \right] \\ - \lim_{Q \rightarrow \infty} \left[\frac{\sqrt{1-k^2}}{\pi} \int_Q^{+1} \left(\eta \frac{a}{t} + \frac{\alpha b}{t+c} + \frac{\gamma b}{t-g} - \frac{d}{t+f} \right) \frac{dt}{\sqrt{t^2-1}} \right] \end{aligned} \quad (15)$$

The details of the integration are shown in Appendix II. The results may be expressed as:

$$\begin{aligned} \frac{a + 2s}{b} = \frac{d}{b} \sin \beta + 2 \frac{\sqrt{1-k^2}}{\pi} \left[\frac{\eta}{k} \frac{a}{b} \sin^{-1} k - \frac{\alpha}{\sqrt{c^2-k^2}} \ln \frac{\sqrt{1-c^2}}{\sqrt{1-k^2} + \sqrt{c^2-k^2}} \right. \\ \left. - \frac{\gamma}{\sqrt{g^2-k^2}} \ln \frac{\sqrt{1-g^2}}{\sqrt{1-k^2} + \sqrt{g^2-k^2}} + \frac{d}{b} \frac{1}{\sqrt{f^2-k^2}} \ln \frac{\sqrt{f^2-1}}{\sqrt{1-k^2} + \sqrt{f^2-k^2}} \right] \end{aligned} \quad (16)$$

Equation (16) establishes the relationship between the various jet velocities and the amplifier setback. Another expression is needed to establish the symmetry of the geometrical configuration. This may be done by performing an integration from D to F along the free-streamline and writing the imaginary part equal to zero so as to ensure that the two sides of the control jets are along the same line and that there is no unsymmetry. Thus, writing

$$\mathcal{J} \left[\int dz \right] = 0 \quad (17)$$

or
$$\mathcal{J} \left[\int_{-1}^{+1} e^{\Omega} \left(-\frac{1}{v_j} \frac{dw}{dt} \right) dt \right] = 0$$

and replacing $\frac{dw}{dt}$ from Eq. (13) and Ω from Eqs. (4) and (6), one has

$$\int_{-1}^{+1} \frac{\sqrt{t^2-1}}{\sqrt{t^2-k^2}} \left(\gamma \frac{a}{\pi} \frac{1}{t} + \frac{\alpha b}{\pi} \frac{1}{t+c} + \frac{\gamma b}{\pi} \frac{1}{t-g} - \frac{d}{\pi} \frac{1}{t+f} \right) dt = 0 \quad (18)$$

Details of the integration are shown in Appendix III. The analysis yields:

$$\begin{aligned} & \frac{2}{\pi (1+k)} \left\{ \left[\left(\frac{1+ck}{k+c} \right) \alpha + \left(\frac{1-gk}{k-g} \right) \gamma - \left(\frac{1+fk}{k+f} \right) \frac{d}{b} \right] \llcorner \left(\frac{2\sqrt{k}}{k+1} \right) \right. \\ & + \frac{\alpha (1-k)(c-1)}{(k+c)} \pi \left[\frac{\pi}{2}, \frac{2(k+c)}{(k+1)(c+1)}, \frac{2\sqrt{k}}{k+1} \right) \\ & + \frac{\gamma (1-k)(1+g)}{(g-k)} \pi \left[\frac{\pi}{2}, \frac{2(k-g)}{(k+1)(1-g)}, \frac{2\sqrt{k}}{k+1} \right) \\ & \left. - \frac{d}{b} \frac{(1-k)(f-1)}{(k+f)} \pi \left[\frac{\pi}{2}, \frac{2(k+f)}{(k+1)(f+1)}, \frac{2\sqrt{k}}{k+1} \right] \right\} = 0 \end{aligned} \quad (19)$$

where $K(h)$ is the complete elliptic integral of the first kind and $\Pi(\frac{\pi}{2}, \alpha^2, h)$ is the complete elliptic integral of the third kind.

Since the parameter α^2 of the elliptic integral of the third kind shown in Eq. (19) was not always in the range of $0 < \alpha^2 < 1$, a subroutine program was written to reduce all the complete elliptic integrals of the third kind into the sum of complete and incomplete elliptic integrals of the first and second kind for which there are tables and subroutines available at the computer facility of the Naval Postgraduate School.

The reductions were made through the use of the transformations given in Ref. [4], and through the use of Heuman's Lambda and Jacobi's Zeta functions, as given in Ref. [5]. The aforementioned reductions are shown in Appendix IV. The evaluations of the resulting expressions for the discontinuity at $t = -f$ is shown in Appendix V. The results obtained for $\frac{S}{a} = 1$ and $\frac{b}{a} = 1$ through the use of the expressions given above will be presented together with the experimental data in connection with the discussion of results.

III. EXPERIMENTAL EQUIPMENT AND PROCEDURE

The experimental apparatus consisted of a jet assembly, hot-wire anemometer system, and a velocity calibrator.

A. JET ASSEMBLY

As shown in Fig. 2 and 3, the entire assembly consisted of three nozzles, one for the power jet and two for the control jets. Each nozzle was $1/4$ inch wide and 2 inches high. This resulted in an aspect ratio of eight. The jet assembly was sandwiched between two $1/4$ inch plexiglass plates.

The width of the control nozzle was adjustable from $1/4$ inch to $1/2$ inch. For all the tests reported here, the control and power jet nozzles, as well as the setback, were maintained at widths of $1/4$ inch.

The smooth streamline contour of the nozzles gave an essentially uniform exit velocity profile. The special configuration adopted for the control jet walls prevented the jet from attaching to the side walls through the Coanda effect. In addition, two large holes, as shown in Fig. 2, were provided in the upper and lower cover-plates, in order to minimize the restriction to entrainment to the resultant jet.

The supply lines were $1-1/4$ inch diameter Tygon tubing, connected to the three inlets. The flow was straightened via a one inch long honey-comb section before undergoing a contraction of ten to one.

Supply pressures were measured by means of inclined manometers through the use of wall pressure tabs located in the straight part of each nozzle.

The top and bottom plates had two slots cut out to enable the hot-wire anemometer mechanism to traverse along lines parallel to the power

nozzle axis. A graph paper, graduated in degrees, was placed under the bottom plate for the purpose of quickly measuring the jet deflection angle by means of a small light string attached to a needle. A more precise measurement of the jet deflection angle was, of course, made by means of the hot-wire anemometer.

Air was supplied to the system from a 50hp compressor at approximately 160 psi. Air first passed through a micro-filter to remove oil and dust and then through the primary pressure regulator. The three jets were independently supplied from this regulator. Each jet supply line had on it a series of pressure regulators, thus providing independently regulated flows into three rotameters. The rotameters were identical and each had a maximum capacity of 17.5 scfm. at 14.7 psia and 70°F. The actual flow rates were obtained from

$$Q = (\text{rotameter reading, \% of full flow}) \times (17.5) \times CF(P)$$

where $CF(P)$ represents the pressure correction factor obtained from a chart provided by the manufacturer.

B. HOT-WIRE ANEMOMETER

A constant temperature hot-wire anemometer, manufactured by the Thermo-Systems, Model 1050, was used in the measurement of velocity and turbulence intensities. The wire used (wire #1272-T1.5) had a cold resistance of 7.30 ohms and an operating resistance of 10.95 ohms. The voltage across the bridge was measured with a digital voltmeter accurate to 0.01 volts.

A special probe holder was constructed and is shown in Fig. 4. This holder which was fitted into the slots in the jet assembly, enabled the probe to move along lines parallel to the power-nozzle axis in the mid-plane. A micrometer mechanism on the holder provided transverse movement of the hot-wire across the jet with an accuracy of 0.001 inch.

The hot-wire probe was calibrated with the Thermo-Systems Calibrator, Model 1125, shown in Fig. 4. An inclined manometer was connected to the reservoir of the calibrator to measure the stagnation pressure and the hot-wire was placed $1/16$ inch above the 0.150 inch orifice of the calibrator. For the details of calibration analysis, see Appendix VI.

IV. EXPERIMENTAL PROCEDURE

The experiments were carried out for a combination of three power jet velocities and for four control jet bias velocities. The bias velocity was defined as $(VC+VG)/2VA$. For each combination, the velocity and turbulence distributions were measured at various distances from the pivoting point, and the results were reduced to actual data through the use of the computerized calibration curve. Most of the experiments were repeated for the purpose of checking the reproducibility of the data. Finally, the deflection angles were determined from the velocity profiles by connecting the peaks of each successive profile with the pivoting point. The deflection angles obtained in this manner did not deviate more than ± 0.5 degrees from those obtained through the use of the floating string and chart paper combination.

The supply pressure for each jet was measured, as previously cited, by means of inclined manometers. The stagnation pressure of the power jet was measured by one manometer, and the stagnation, as well as differential pressure of the control jets, through the use of two other manometers.

V. DISCUSSION OF RESULTS

In the following, first the experimental results obtained with the hot-wire anemometer will be discussed and then a comparison will be made between the theoretically and experimentally obtained deflection angle.

Figures 6 through 54 represent the velocity and turbulence intensity distributions as a function of x/a and U/V_{RO} in terms of normalized control jet velocities, deflection angle, and the distance from the pivoting point. It is apparent from all of these figures that the control jets retain their identity up to a distance of $2a$ from the pivoting point. This region will be called the mixing region. For distances larger than approximately $2a$, the velocity profiles acquire a nearly Gaussian distribution. The turbulence intensity distribution exhibits essentially four peaks: two in the outer edges of the resultant jet because of the mixing of the jet with the ambient flow and two peaks in the regions where the control jets mix with the power jet. As the distances increase from the pivoting point, the turbulence intensity in the inner regions decreases and eventually reduces to a single minimum at the axis of the jet. It is therefore apparent that the application of the free-streamline theory to the mixing of jets of unequal velocities is not entirely valid in the mixing region since there are three independent free-streamline velocities. In the fully established region, the velocity profile is Gaussian and thus far from being uniform. Consequently, the use of the single jet velocity is not fully justified unless special precautions are taken to make sure that the gross continuity of the mass flow is satisfied by properly selecting the apparent jet width. In so doing, the maximum centerline velocity may be used as the free-streamline

velocity and the effective jet velocity may be calculated from the equation of continuity. The theoretical analysis presented herein makes good use of this fact and shows that the results so obtained are in good agreement with those obtained experimentally.

Figures 55 through 57 show a superposition of the velocity profiles and the evolution of the mixing from three independent jets to a single profile. Similar graphs are presented in Figs. 58 and 59 for the turbulence intensity. Figure 60 is a three-dimensional plot of the evolution of the velocity profiles as a function of the distance from the pivoting point.

The deflection angle is shown in Figs. 61 through 63 as a function of the sum and difference of the normalized control jets for all Reynolds numbers. It is immediately apparent that the deflection angle is a linear function of the normalized differential control jets velocity and that the deflection angle does not vary appreciably with the normalized average of the control jets. These results suggest that the characteristics of beam-deflection amplifiers be evaluated in terms of the parameters cited above rather than those previously used by other investigators. Although the experimental results presented herein are for equal control and power jet nozzles, there is every reason to believe that similar correlations will exist between the jet deflection angle and the two control jets parameters.

A comparison of the experimental results with those obtained theoretically for a normalized average control jet velocity of unity is shown in Fig. 64. As previously cited, the final jet width d used in the calculations was obtained by dividing the total flow rate by the maximum jet velocity. It is believed that there is at present no other

way of applying the free-streamline theory to the deflection of viscous jets for in the mixing region there are three different jet velocities and three fairly uniform velocity distributions and, in the fully established region, a single maximum with a Gaussian distribution. Thus, the only logical assumption to be made is that the free-streamline velocity used in the analysis is equal to the maximum of the fully established profile and that the effective jet width is determined by the equation of continuity. In reality, it appears that the maximum centerline velocity is somewhat higher than that predicted by the free-streamline theory and that the effective jet width is somewhat smaller than that used in the analysis. Be that as it may, no second-order corrections were applied to the analysis in order to preserve its generality. The extension of the analysis to other velocity ratios will be separately undertaken in the future.

The stagnation pressures at supply reservoirs of the three jets were determined as previously explained and are plotted in Fig. 65. Mean values rather than the actual data points are shown in this graph because the data points fell fairly consistently on straight lines. It is immediately apparent that the normalized differential control port pressures are linear functions of the deflection angle and depend weakly on the mean pressure of the control ports. A simple analysis of the jet deflection through the use of the momentum balance equation shows that the deflection angle so determined is considerably higher than that obtained experimentally or theoretically even when the pressure differences between various nozzles are taken into consideration.

VI. CONCLUSIONS

The foregoing theoretical and experimental study warrants the following conclusions:

- (a) The jets preserve their identity in a region of approximately $2a$ from the pivoting point;
- (b) The velocity profiles become Gaussian beyond the mixing region;
- (c) The free streamline theory predicts the deflection angles fairly accurately provided that an effective jet width (calculated in terms of the mass flow and the maximum centerline velocity) is incorporated into the analysis;
- (d) The previous models cited in the introduction do not predict, or are not capable of predicting, the jet deflection angle to an accuracy demanded by the beam deflection amplifiers; and finally,
- (e) The normalized differential control reservoir pressures vary linearly with the deflection angle. It is thus believed that an analysis such as the one presented here, together with one experimentally determined parameter, is fully capable of predicting the performance of proportional amplifiers. The knowledge of the jet deflection angle and the Gaussian shape of the velocity profile provide through a simple integration the determination of the amounts of flow at a given amplifier receiver.

APPENDIX I

EVALUATION OF THE INTEGRAL

$$\Omega = \int \frac{M t dt}{(t^2 - k^2) \sqrt{t^2 - 1}} + N \quad (20)$$

Letting $x^2 = t^2 - k^2$

$$x dx = t dt$$

one has

$$\Omega = M \int \frac{dx}{x \sqrt{x^2 + (k^2 - 1)}} + N$$

The general solution of this integral, irrespective of whether the radicals are real or imaginary, is given in Gröbner 233.3b [Ref. 6] as,

$$\Omega = - \frac{M}{\sqrt{k^2 - 1}} \ln \frac{(k^2 - 1) + \sqrt{(k^2 - 1)(x^2 + (k^2 - 1))}}{x} + N \quad (21)$$

replacing t and rearranging, we have

$$\Omega = - \frac{M}{i \sqrt{k^2 - 1}} \ln \frac{(k^2 - 1) + \sqrt{k^2 - 1} \sqrt{t^2 - 1}}{\sqrt{t^2 - k^2}} + N \quad (22)$$

or

$$\Omega = - \frac{M}{i \sqrt{k^2 - 1}} \ln \frac{\sqrt{k^2 - 1} + \sqrt{t^2 - 1}}{\sqrt{t^2 - k^2}} + N_1 \quad (23)$$

or

$$\Omega = - \frac{M}{i \sqrt{k^2 - 1}} \ln \frac{\sqrt{1 - k^2} + i \sqrt{t^2 - 1}}{\sqrt{t^2 - k^2}} + N_2 \quad (24)$$

or

$$\Omega = M_2 \ln \frac{\sqrt{1 - k^2} + i \sqrt{t^2 - 1}}{\sqrt{t^2 - k^2}} + N_2 \quad (25)$$

The constants M_2 and N_2 may be determined, for the different regions of the t plane, as follows:

a. At the point D, $t = -1$, $\frac{V_j}{q} = 1$, $\theta = 0$

i) approaching from left to D $t = -1 - \varepsilon$

$$\begin{aligned} 0 + i0 &= \lim_{\varepsilon \rightarrow 0} \left[\frac{-M}{i\sqrt{1-k^2}} \ln \frac{\sqrt{1-k^2} + i\sqrt{(-1-\varepsilon)^2 - 1}}{\sqrt{(-1-\varepsilon)^2 - k^2}} \right] + N_2 \\ &= M_2 \ln \frac{\sqrt{1-k^2}}{\sqrt{1-k^2}} + N_2 \end{aligned}$$

Then $N_2 = 0$

ii) approaching from right to D $t = -1 + \varepsilon$

$$\begin{aligned} 0 + i0 &= \lim_{\varepsilon \rightarrow 0} \left[\frac{-M}{i\sqrt{1-k^2}} \ln \frac{\sqrt{1-k^2} + i\sqrt{(-1+\varepsilon)^2 - 1}}{\sqrt{(-1+\varepsilon)^2 - k^2}} \right] + N_2 \\ &= \frac{-M}{i\sqrt{1-k^2}} \ln \frac{\sqrt{1-k^2}}{\sqrt{1-k^2}} + N_2 \end{aligned}$$

Then $N_2 = 0$

Furthermore, since at $t = -\infty$, $\theta = \frac{\pi}{2}$

one has $0 + i\frac{\pi}{2} = \frac{-M}{i\sqrt{1-k^2}} \ln(0 + i)$

Then

$$M_2 = \frac{-M}{i\sqrt{1-k^2}} = 1$$

b. At the point B, $t = -k$, $\frac{V_i}{q} = \infty$

i) approaching from left to B, $t = 0$, $t = -k - \epsilon$

$$\infty + i0 = M_2 \lim_{\epsilon \rightarrow 0} \left[\ln \frac{\sqrt{1-k^2} + i\sqrt{(-k-\epsilon)^2 - 1}}{\sqrt{(-k-\epsilon)^2 - k^2}} \right] + N_2$$

$$\infty + i0 = M_2 \lim_{\epsilon \rightarrow 0} \left[\ln \frac{2\sqrt{1-k^2}}{\sqrt{2k\epsilon + \epsilon^2}} \right] + N_2$$

$$\infty + i0 = M_2 \infty + N_2$$

Thus $N_2 = 0$

and $M_2 = 1$

ii) approaching from right to B, $\theta = \frac{\pi}{2}$, $t = -k + \epsilon$

$$\infty + i\frac{\pi}{2} = M_2 \lim_{\epsilon \rightarrow 0} \left[\ln \frac{\sqrt{1-k^2} + i\sqrt{(-k+\epsilon)^2 - 1}}{\sqrt{(-k+\epsilon)^2 - k^2}} \right] + N_2$$

$$\infty + i\frac{\pi}{2} = M_2 \lim_{\epsilon \rightarrow 0} \left[\ln \frac{2\sqrt{1-k^2}}{\sqrt{-2\epsilon k}} \right] + N_2$$

$$\infty + i\frac{\pi}{2} = M_2 \lim_{\epsilon \rightarrow 0} \left[\ln \frac{2\sqrt{1-k^2}}{\epsilon\sqrt{-2k}} \right] + N_2$$

$$\infty + i\frac{\pi}{2} = M_2 \left[\ln \infty - i\frac{\pi}{2} \right] + N_2$$

Thus $M_2 = 1$ and $N_2 = i\pi$

c. At the point H,

$$t = +k, \quad \frac{V_1}{q} = \infty$$

i) approaching from left, $\theta = \frac{\pi}{2}, \quad t = k - \varepsilon$

$$\infty + i\frac{\pi}{2} = M_2 \lim_{\varepsilon \rightarrow 0} \left[\ln \frac{\sqrt{1-k^2} + i\sqrt{(k-\varepsilon)^2 - 1}}{\sqrt{(k-\varepsilon)^2 - k^2}} \right] + N_2$$

$$\infty + i\frac{\pi}{2} = M_2 \lim_{\varepsilon \rightarrow 0} \left[\ln \frac{2\sqrt{1-k^2}}{i\sqrt{2\varepsilon k}} \right] + N_2$$

$$\infty + i\frac{\pi}{2} = M_2 \left[\ln \infty - i\frac{\pi}{2} \right] + N_2$$

Thus

$$M_2 = 1 \quad \text{and} \quad N_2 = i\pi$$

ii) approaching from right, $\theta = \pi, \quad t = k + \varepsilon$

$$\infty + i\pi = M_2 \lim_{\varepsilon \rightarrow 0} \left[\ln \frac{\sqrt{1-k^2} + i\sqrt{(k+\varepsilon)^2 - 1}}{\sqrt{(k+\varepsilon)^2 - k^2}} \right] + N_2$$

$$\infty + i\pi = M_2 \lim_{\varepsilon \rightarrow 0} \left[\ln \frac{2\sqrt{1-k^2}}{\sqrt{2k\varepsilon}} \right] + N_2$$

$$\infty + i\pi = M_2 \ln \infty + N_2$$

Thus

$$M_2 = 1 \quad \text{and} \quad N_2 = i\pi$$

d. At the point F,

$$t = +1, \quad \theta = \pi, \quad \frac{v_2}{t} = 1$$

i) Where approaching from left

$$t = 1 - \varepsilon$$

$$0 + i\pi = M_2 \lim_{\varepsilon \rightarrow 0} \left[\ln \frac{\sqrt{1-k^2} + i\sqrt{(1-\varepsilon)^2 - k^2}}{\sqrt{(1-\varepsilon)^2 - k^2}} \right] + N_2$$

$$0 + i\pi = M_2 \lim_{\varepsilon \rightarrow 0} \left[\ln \frac{\sqrt{1-k^2} + i\varepsilon}{\sqrt{1-k^2}} \right] + N_2$$

Thus

$$M_2 = 1$$

and

$$N_2 = i\pi$$

ii) approaching from right,

$$t = 1 + \varepsilon$$

$$0 + i\pi = M_2 \lim_{\varepsilon \rightarrow 0} \left[\ln \frac{\sqrt{1-k^2} + i\sqrt{(1+\varepsilon)^2 - k^2}}{\sqrt{(1+\varepsilon)^2 - k^2}} \right] + N_2$$

$$0 + i\pi = M_2 \lim_{\varepsilon \rightarrow 0} \left[\ln \frac{\sqrt{1-k^2} + i\sqrt{\varepsilon^2}}{\sqrt{(1+\varepsilon)^2 - k^2}} \right] + N_2$$

$$0 + i\pi = M_2 \ln \frac{\sqrt{1-k^2}}{\sqrt{1-k^2}} + N_2$$

Thus

$$M_2 = -1$$

and

$$N_2 = i\pi$$

Furthermore, where

$$\text{as } t \rightarrow +\infty, \quad \theta = \frac{\pi}{2} \quad \text{or} \quad \lim_{\varepsilon \rightarrow \infty}$$

$$0 + i\frac{\pi}{2} = M_2 \ln(0+i) + N_2$$

$$0 + i\frac{\pi}{2} = M_2 \left(i\frac{\pi}{2} \right) + N_2$$

Thus

$$M_2 = -1$$

and

$$N_2 = i\pi$$

Finally, Ω reduces to

$$\Omega = \ln \frac{\sqrt{1-k^2} + i\sqrt{t^2-1}}{\sqrt{t^2-1}} \quad \text{for } t \leq -k \quad (26)$$

$$\Omega = \ln \frac{\sqrt{1-k^2} + i\sqrt{t^2-1}}{-\sqrt{t^2-k^2}} \quad \text{for } -k \leq t \leq 1 \quad (27)$$

$$\Omega = -\ln \frac{\sqrt{1-k^2} + i\sqrt{t^2-1}}{-\sqrt{t^2-k^2}} \quad \text{for } 1 \leq t \quad (28)$$

Evidently, Eq. (28) is equivalent to

$$\Omega = \ln \frac{-\sqrt{1-k^2} + i\sqrt{t^2-1}}{\sqrt{t^2-k^2}} \quad (29)$$

APPENDIX II

EVALUATION OF THE INTEGRAL

$$\int \frac{\sqrt{1-k^2}}{\pi} \left[\frac{a}{k} + \frac{\alpha b}{t+c} + \frac{\beta b}{t-g} - \frac{d}{t+f} \right] \frac{dt}{\sqrt{t^2-k^2}} \quad (30)$$

Working each term of the integral separately, one has

A1. From CRC Table, formula - 134

$$\int \frac{dt}{t \sqrt{t^2-k^2}} = \frac{1}{k} \sec^{-1} \frac{t}{k} + C = \frac{1}{k} \cos^{-1} \frac{k}{t} + C \quad (31)$$

A2. From Gröbner and Hofreiter [Ref. 6] formula - 231.10a

$$\begin{aligned} \int \frac{dt}{(t+c) \sqrt{t^2-k^2}} &= -\frac{1}{\sqrt{c^2-k^2}} \ln \frac{\sqrt{(t^2-k^2)(c^2-k^2)}-k^2-ct}{(t+c)\sqrt{c^2-k^2}} + C_1 \\ &= -\frac{1}{\sqrt{c^2-k^2}} \ln \frac{\sqrt{(t^2-k^2)(c^2-k^2)}-k^2-ct}{t+c} + C_2 \quad (32) \end{aligned}$$

A3. Using the same method as A2

$$\int \frac{dt}{(t-g) \sqrt{t^2-k^2}} = -\frac{1}{\sqrt{g^2-k^2}} \ln \frac{\sqrt{(t^2-k^2)(g^2-k^2)}-k^2-gt}{t-g} + C_3 \quad (33)$$

A4. Using the same method as A2

$$\int \frac{dt}{(t-f) \sqrt{t^2-k^2}} = -\frac{1}{\sqrt{f^2-k^2}} \ln \frac{\sqrt{(t^2-k^2)(f^2-k^2)}-k^2-ft}{t+f} + C_4 \quad (34)$$

LIMIT PROCESS

Equations (31), (32) and (33) have no discontinuity at $t = -f$

For Eq. (31), one has

$$\frac{1}{k} \left[\left(\cos^{-1} \frac{k}{-\infty} - \cos^{-1} \frac{k}{-1} \right) - \left(\cos^{-1} \frac{k}{+1} - \cos^{-1} \frac{k}{+\infty} \right) \right]$$

which reduces to

$$\frac{2}{k} \sin^{-1} k \quad (35)$$

For Eq. (32) we have

$$\lim_{Q \rightarrow \infty} \left[-\frac{1}{\sqrt{c^2 - k^2}} \ln \frac{(\sqrt{(Q^2 - k^2)(c^2 - k^2)} - k^2 + cQ)(\sqrt{(Q^2 - k^2)(c^2 - k^2)} - k^2 - cQ)(c+1)(c-1)}{(c-Q)(c+Q)(\sqrt{(1-k^2)(c^2 - k^2)} - k^2 + c)(\sqrt{(1-k^2)(c^2 - k^2)} - k^2 - c)} \right]$$

which reduces to

$$-\frac{2}{\sqrt{c^2 - k^2}} \ln \frac{\sqrt{1 - c^2}}{\sqrt{1 - k^2} + \sqrt{c^2 - k^2}} \quad (36)$$

For Eq. (33), using the same procedure as above, one has

$$-\frac{2}{\sqrt{g^2 - k^2}} \ln \frac{\sqrt{1 - g^2}}{\sqrt{1 - k^2} + \sqrt{g^2 - k^2}} \quad (37)$$

For Eq. (34) the inclusion of the discontinuity at $t = -f$ yields

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ Q \rightarrow \infty}} \left\{ -\frac{1}{\sqrt{f^2 - k^2}} \ln \frac{[\sqrt{(f^2 - k^2)(f^2 - k^2)} - k^2 + f(f - \varepsilon)][\sqrt{(Q^2 - k^2)(f^2 - k^2)} - k^2 + fQ]}{(f - f + \varepsilon)(f - Q)(f + Q)[\sqrt{(1 - k^2)(f^2 - k^2)} - k^2 + f]} \right. \\ \left. \cdot \frac{[\sqrt{(Q^2 - k^2)(f^2 - k^2)} - k^2 - fQ](f - 1)(f - f - \varepsilon)(f + 1)}{[\sqrt{(f^2 - k^2)(f^2 - k^2)} - k^2 + f(f + c)][\sqrt{(1 - k^2)(f^2 - k^2)} - k^2 - f]} \right\}$$

which after simplification reduces to

$$-\frac{2}{\sqrt{f^2 - k^2}} \ln \frac{\sqrt{f^2 - 1}}{\sqrt{1 - k^2} + \sqrt{f^2 - k^2}} \quad (38)$$

Replacing Eqs. (35), (36), (37) and (38) in Eq. (15) one has

$$a + 2s = d \sin \beta + \frac{2\sqrt{1-k^2}}{\pi} \left[\frac{\gamma a}{k} \sin^{-1} k - \frac{\gamma b}{\sqrt{c^2-k^2}} \ln \frac{\sqrt{1-c^2}}{\sqrt{1-k^2} + \sqrt{c^2-k^2}} \right. \\ \left. - \frac{\gamma b}{\sqrt{g^2-k^2}} \ln \frac{\sqrt{1-g^2}}{\sqrt{1-k^2} + \sqrt{g^2-k^2}} + \frac{d}{\sqrt{f^2-k^2}} \ln \frac{\sqrt{f^2-1}}{\sqrt{1-k^2} + \sqrt{f^2-k^2}} \right] \quad (39)$$

APPENDIX III

EVALUATION OF THE INTEGRAL

$$\int_{-1}^{+1} \sqrt{\frac{t^2-1}{t^2-k^2}} \left(\gamma \frac{a}{\pi} \frac{1}{t} + \frac{\alpha b}{\pi} \frac{1}{t+c} + \frac{\gamma b}{\pi} \frac{1}{t-g} - \frac{d}{\pi} \frac{1}{t+f} \right) dt \quad (40)$$

Consider the following typical integral

$$\int_{-1}^{+1} \sqrt{\frac{t^2-1}{t^2-k^2}} \frac{dt}{t+p} \quad (41)$$

Multiplying by $\sqrt{t^2-1}$ the numerator and the denominator, we have

$$\int_{-1}^{+1} \frac{(t^2-1) dt}{\sqrt{(t^2-1)(t^2-k^2)} \cdot (t+p)}$$

Dividing (t^2-1) by $(t+p)$, we can express the integral as

$$\int_{-1}^{+1} \frac{t dt}{\sqrt{(t^2-1)(t^2-k^2)}} - p \int_{-1}^{+1} \frac{dt}{\sqrt{(t^2-1)(t^2-k^2)}} + (p^2-1) \int_{-1}^{+1} \frac{dt}{(t+p) \sqrt{(t^2-k^2)(t^2-1)}} \quad (42)$$

Note that the first integral in Eq. (42) is an odd function of t . Thus its integration from -1 to $+1$ reduces to zero. The same result is obtained for the third integral when $p = 0$.

From Gröbner [Ref. 6] formulas 244.8a 4 and 244.8a 7 one has

$$\int \frac{dt}{\sqrt{(t^2-1)(t^2-k^2)}} = \frac{2}{1+k} F(\varphi, h) \quad (43)$$

and

$$\int \frac{dt}{(t+p) \sqrt{(t^2-1)(t^2-k^2)}} = \frac{1}{(k+p)(k+1)} \left\{ 2 F(\varphi, h) + \frac{2(k-1)}{(1+p)} \Pi \left(\varphi, \frac{2(k+p)}{(k+1)(p+1)}, h \right) \right\} \quad (44)$$

where

$$h = \frac{2\sqrt{k}}{1+k} \quad (45)$$

and ψ is given by

$$\sin^2 \psi = \frac{(k+1)(t-1)}{2(t-k)} \quad (46)$$

and $F(\psi, h)$ is defined by
$$\int_0^{\sin \psi} \frac{dx}{\sqrt{(1-x^2)(1-h^2x^2)}}$$

and $\Pi(\psi, \alpha^2, h)$ is defined as
$$\int_0^{\sin \psi} \frac{dx}{(1-\alpha^2x^2)\sqrt{(1-x^2)(1-h^2x^2)}}$$

In taking the limit of Eqs. (43) and (44), it must be noted that the integral from -1 to +1 is to be performed by going from -1 to $-\infty$, to $+\infty$ to +1. Replacing the value of t in Eq. (46), it is seen that ψ goes from $\frac{\pi}{2}$ to zero when t goes from -1 to +1, in the sense described above. Replacing Eqs. (43) and (44) in Eq. (42) and noting that ψ goes from $\frac{\pi}{2}$ to zero, we have:

$$\int_{-1}^{+1} \sqrt{\frac{t^2-1}{t^2-k^2}} \frac{dt}{t+p} = \frac{2}{1+k} \left\{ \frac{1+pk}{k+p} K(h) + \frac{(1-k)(p-1)}{(k+p)} \Pi\left(\frac{\pi}{2}, \frac{2(k+p)}{(k+1)(p+1)}, h\right) \right\} \quad (47)$$

where $K(h)$ is the complete elliptic integral of the first kind and h is as defined by Eq. (45).

Replacing Eq. (47) in Eq. (40) and replacing p by its appropriate value, we have

$$\begin{aligned}
 & \int_{-1}^{+1} \sqrt{\frac{t^2-1}{t^2-k^2}} \left(\gamma \frac{a}{\pi} \frac{1}{t} + \frac{\alpha b}{\pi} \frac{1}{t+c} + \frac{\gamma b}{\pi} \frac{1}{t-g} - \frac{d}{\pi} \frac{1}{t+f} \right) dt = \\
 & \frac{2b}{\pi(1+k)} \left\{ \left[\left(\frac{1+ck}{k+c} \right) \alpha + \left(\frac{1-gk}{k-g} \right) \gamma - \left(\frac{1+fk}{k+f} \right) \frac{d}{b} \right] K(h) \right. \\
 & + \frac{\alpha(1-k)(c-1)}{(k+c)} \Pi \left(\frac{\pi}{2}, \frac{2(k+c)}{(k+1)(c+1)}, h \right) \\
 & + \frac{\gamma(1-k)(1+g)}{(g-k)} \Pi \left(\frac{\pi}{2}, \frac{2(k-g)}{(k+1)(1-g)}, h \right) \\
 & \left. - \frac{d}{b} \frac{(1-k)(f-1)}{(k+f)} \Pi \left(\frac{\pi}{2}, \frac{2(k+f)}{(k+1)(f+1)}, h \right) \right\} \quad (48)
 \end{aligned}$$

The singularity at $t = -f$ is evaluated separately in Appendix V.

APPENDIX IV

EVALUATION OF COMPLETE ELLIPTIC INTEGRAL OF THE THIRD KIND

In the general case of Complete Elliptic Integrals of third kind

$$\Pi\left(\frac{\pi}{2}, \alpha^2, h\right) \quad (49)$$

According to C. Heuman [Ref. 5] they can be reduced to combinations of first and second kind elliptic integrals, depending on the value of α^2 , that is depending on whether the integrals are hyperbolic or circular.

In the present case

$$\alpha^2 = \frac{2(k+p)}{(k+1)(1+p)} \quad (50)$$

where p can be f , c , or $-g$ and

$$h = \frac{2\sqrt{h}}{k+1} \quad (51)$$

a. For the case of $p = f$, we have $\alpha^2 > 1$

$$\Pi\left(\frac{\pi}{2}, \alpha^2, h\right) = - \frac{\alpha K Z(\beta, h)}{\sqrt{(\alpha^2-1)(\alpha^2-h^2)}} \quad (52)$$

where

$$\beta = \sin^{-1}\left(\frac{1}{\alpha}\right)$$

b. For the case of $p = c$, we have $h^2 < \alpha^2 < 1$

and the transformation becomes

$$\Pi\left(\frac{\pi}{2}, \alpha^2, h\right) = \frac{\pi}{2} \frac{\alpha \Lambda_0\left(\frac{\pi}{2}, h\right)}{\sqrt{(\alpha^2-h^2)(1-\alpha^2)}} \quad (53)$$

where

$$\xi = \sin^{-1} \sqrt{\frac{\alpha^2 - h^2}{\alpha^2(1-h^2)}}$$

c. For the case of $p = -g$, we have $\alpha^2 < 0$

and the transformation becomes:

$$\Pi\left(\frac{\pi}{2}, \alpha^2, h\right) = \frac{h^2 K}{h^2 - \alpha^2} - \frac{\pi}{2} \frac{\alpha^2 \Lambda_0(\phi, h)}{\sqrt{\alpha^2(1-\alpha^2)(\alpha^2 - h^2)}} \quad (54)$$

where

$$\phi = \sin^{-1} \sqrt{\frac{\alpha^2}{\alpha^2 - h^2}}$$

where the Heuman's Lambda function Λ_0 is defined as

$$\Lambda_0(\beta, h) = \frac{2}{\pi} \left[E F(\beta, h') + K E(\beta, h') - K F(\beta, h') \right] \quad (55)$$

and the Jacobi's Zeta function Z is defined as

$$Z(\beta, h) = E(\beta, h) - \frac{E}{K} F(\beta, h) \quad (56)$$

where

$$E(\beta, h) \text{ is the elliptic integral of the second kind} \quad (57)$$

$$F(\beta, h) \text{ is the elliptic integral of the first kind} \quad (58)$$

$$\text{and } E = E\left(\frac{\pi}{2}, h\right) \quad (59)$$

$$K = F\left(\frac{\pi}{2}, h\right) \quad (60)$$

$$\text{and } h'^2 = 1 - h^2$$

Subroutine CEL3 was written to calculate the complete elliptic integrals of the third kind for all the cases of α^2 cited above.

APPENDIX V

EVALUATION OF THE DISCONTINUITY OF THE INTEGRAL

$$\int_{-1}^{+1} \frac{dt}{(t+f) \sqrt{(t^2-k^2)(t^2-1)}} \quad (61)$$

Since there is a discontinuity at $t = -f$, we have to go to the limiting process as follows

$$\int_{-1}^{+1} f(t) = \int_{-1}^{-f+\varepsilon} f(t) + \int_{-f-\varepsilon}^{+1} f(t) \quad \text{as } \varepsilon \rightarrow 0 \quad (62)$$

As calculated in Appendix IV

$$\int \frac{dt}{(t+f) \sqrt{(t^2-k^2)(t^2-1)}} = \frac{2}{(1+k)(f+k)} \left\{ F(\varphi, h) - \frac{(1-k)}{(f+1)} \Pi\left(\varphi, \frac{2(k+f)}{(1+k)(1+f)}, h\right) \right\} \quad (63)$$

where

$$\sin^2 \varphi = \frac{(t-1)(1+k)}{2(t-k)} \quad (64)$$

When taking $\int_{-1}^{+1} f(t) dt$, we have $\int_{-1}^{+1} f(t) dt$ which means that $F(\varphi, h)$ reduces to K (complete elliptic integral of the first kind) and $\Pi(\varphi, \alpha^2, h)$ reduces to $\Pi(\alpha^2, h)$ (complete elliptic integral of the third kind). The only term which has a discontinuity is $\Pi(\varphi, \alpha^2, h)$ since $F(\varphi, h)$ is continuous.

At $t = -f + \varepsilon$

$$\varphi_1 = \sin^{-1} \sqrt{\frac{(1+k)(1+f-\varepsilon)}{2(f+k-\varepsilon)}}$$

and at $t = -f - \varepsilon$

$$\varphi_2 = \sin^{-1} \sqrt{\frac{(1+k)(1+f+\varepsilon)}{2(f+k+\varepsilon)}}$$

Then Eq. (62) becomes

$$B(0) - B(\varphi_2) + B(\varphi_1) - B\left(\frac{\pi}{2}\right) \quad (65)$$

Let us examine in particular, $B(\varphi_1) - B(\varphi_2)$, or

$$\mathcal{T}(\varphi_1, \alpha^2, h) - \mathcal{T}(\varphi_2, \alpha^2, h) \quad (66)$$

Transforming $\mathcal{T}(\varphi, \alpha^2, h)$ into a combination of first kind elliptic integrals and Zeta and Theta functions, according to [Ref. 4], we have

$$\mathcal{T}(\varphi, \alpha^2, h) = - \frac{\alpha [F(\varphi, h) Z(A, h) - \Omega_4]}{\sqrt{(\alpha^2 - 1)(\alpha^2 - h^2)}} \quad (67)$$

where

$$\Omega_4 = \frac{1}{2} \ln \frac{\sin[\omega(A) + v]}{\sin[\omega(A) - v]} + \sum_{m=1}^{\infty} q^m \frac{\sin 2mv \sin 2m\omega(A)}{m \sinh 2mp}$$

$$A = \sin^{-1}\left(\frac{1}{\alpha}\right)$$

$$v = \frac{\pi}{2K} F(\varphi, h)$$

$$p = \frac{\pi}{2K} K(h')$$

$$\omega(\xi) = \frac{\pi}{2K} F(\xi, h)$$

$$q = e^{-2p}$$

when we take $B(\varphi_1) - B(\varphi_2)$ where $B(\varphi)$ is given by Eq. (67), we are

left with

$$B(\varphi_1) - B(\varphi_2) = - \frac{\alpha [\Omega_4(\varphi_1) - \Omega_4(\varphi_2)]}{\sqrt{(\alpha^2 - 1)(\alpha^2 - h^2)}} \quad (68)$$

Examining Ω_4 further we find that every term is independent of ε except v , which is a function of φ . Thus Eq. (68) may be written as

$$B(\varphi_1) - B(\varphi_2) = - \frac{\alpha}{2\sqrt{(\alpha^2 - 1)(\alpha^2 - h^2)}} \left[\ln \frac{\sin \left[\frac{\pi}{2K} \{F(A, h) + F(\varphi_1, h)\} \right]}{\sin \left[\frac{\pi}{2K} \{F(A, h) - F(\varphi_1, h)\} \right]} - \ln \frac{\sin \left[\frac{\pi}{2K} \{F(A, h) + F(\varphi_2, h)\} \right]}{\sin \left[\frac{\pi}{2K} \{F(A, h) - F(\varphi_2, h)\} \right]} \right] \quad (69)$$

Simplifying, Eq. (69) reduces to

$$= - \frac{\frac{x}{2}}{\sqrt{(\alpha^2-1)(\alpha^2-1)}} \ln \frac{\sin \left[\frac{\pi}{2K} \left\{ F(A, h) + F(\psi_1, h) \right\} \right] \cdot \sin \left[\frac{\pi}{2K} \left\{ F(A, h) - F(\psi_2, h) \right\} \right]}{\sin \left[\frac{\pi}{2K} \left\{ F(A, h) - F(\psi_1, h) \right\} \right] \cdot \sin \left[\frac{\pi}{2K} \left\{ F(A, h) + F(\psi_2, h) \right\} \right]}$$

in the limit as $\varepsilon \rightarrow 0$, $\psi_1 \rightarrow \psi_2 \rightarrow \psi$

The ratio inside the \ln will reduce to unity unless $F(A, h) = \pm F(\psi, h)$

Replacing A by

$$A = \sin^{-1} \frac{1}{\alpha} = \sin^{-1} \sqrt{\frac{(1+k)(1+f)}{2(k+f)}}$$

and ψ by

$$\psi = \sin^{-1} \sqrt{\frac{(1+k)(1+f)}{2(k+f)}}$$

Then $A = \psi$

Thus we have an undetermined case of the type $\ln \frac{0}{0}$

Defining

$$\begin{aligned} F(\psi_1, h) &= F_1 \\ F(\psi_2, h) &= F_2 \\ F(A, h) &= F \\ \frac{\pi}{2K} &= a \\ - \frac{\alpha/2}{\sqrt{(\alpha^2-1)(\alpha^2-h^2)}} &= b \end{aligned}$$

and using L'Hopital's rule, we have

$$= \lim_{\varepsilon \rightarrow 0} b \ln \frac{\sin(aF + aF_1)}{\sin(aF + aF_2)} + \lim_{\varepsilon \rightarrow 0} b \ln \frac{\sin(aF - aF_2)}{\sin(aF - aF_1)} \quad (70)$$

which is equal to

$$= \lim_{\varepsilon \rightarrow 0} b \ln \frac{\sin(aF - aF_2)}{\sin(aF - aF_1)}$$

Using L'Hopital's rule

$$= \lim_{\varepsilon \rightarrow 0} b \ln \frac{\cos(aF - aF_2) \frac{\partial F_2}{\partial \varepsilon}}{\cos(aF - aF_1) \frac{\partial F_1}{\partial \varepsilon}}$$

or

$$= \lim_{\varepsilon \rightarrow 0} b \ln \frac{\frac{\partial F_2}{\partial \varepsilon}}{\frac{\partial F_1}{\partial \varepsilon}} \quad (71)$$

From Hildebrand (Adv. Calc. for Eng.) page 352, one has

$$\frac{\partial F(\varphi, h)}{\partial \varepsilon} = f(\sin \varphi) \frac{\partial \sin \varphi}{\partial \varepsilon} \quad (72)$$

where

$$f(\sin \varphi_1) = f(\varepsilon) = \frac{1}{\sqrt{(1-k^2) \left(1 - \frac{k^2(1+k)(1+f+\varepsilon)}{2(f+k+\varepsilon)}\right)}} \quad (73)$$

Then

$$\frac{\partial \sin(\varphi_1)}{\partial \varepsilon} = \sqrt{\frac{(k+1)}{2}} \frac{(k-1)}{2(f+k+\varepsilon)^{3/2} \sqrt{f+1+\varepsilon}} \quad (74)$$

and

$$\frac{\partial \sin(\varphi_2)}{\partial \varepsilon} = -\sqrt{\frac{(k+1)}{2}} \frac{(1-k)}{2(f+k-\varepsilon)^{3/2} \sqrt{f+1-\varepsilon}} \quad (75)$$

Now replacing Eqs. (72), (73), (74) and (75) in Eq. (71), and letting

$\varepsilon \rightarrow 0$, one has

$$B(\varphi_1) - B(\varphi_2) = 0$$

APPENDIX VI

DERIVATION OF TURBULENCE - INTENSITY EQUATION

On a hot-wire anemometer the heat transfer equilibrium equation is given by

$$\frac{I^2 R_w}{R_w - R_g} = A + B \sqrt{U} \quad (76)$$

where I = current through the wire

R_w = operating resistance

R_g = cold resistance

A and B = constants

U = instantaneous velocity of the fluid

From (76) at $U = 0$ $I = I_0$

So

$$\frac{I^2 R_w}{R_w - R_g} = \frac{I_0^2 R_w}{R_w - R_g} + B \sqrt{U}$$

Multiplying both sides of the equation by R_w and replacing $I^2 R^2$ by E^2 we obtain

$$E^2 = E_0^2 + B R_w (R_w - R_g) \sqrt{U}$$

or

$$E^2 = E_0^2 + C \sqrt{U} \quad (77)$$

Since the wire is kept at a constant temperature R_w and R_g are constants.

Taking the derivative with respect to time, one has

$$2E \frac{dE}{dt} = \frac{C \sqrt{U}}{2U} \frac{dU}{dt} \quad (78)$$

From Eq. (77) $c\sqrt{U} = E^2 - E_0^2$

Then Eq. (78) becomes

$$\frac{1}{U} \frac{dU}{dt} = \frac{4E}{E^2 - E_0^2} \frac{dE}{dt}$$

where $\frac{dU}{dt} = \sqrt{\bar{u}^2}$ = velocity variation

and $\frac{dE}{dt} = \sqrt{\bar{e}^2}$ = voltage variation

Finally we have

$$\frac{\sqrt{\bar{u}^2}}{U} = \frac{4E}{E^2 - E_0^2} \sqrt{\bar{e}^2}$$

in other words, turbulence intensity is equal to $\frac{4E}{E^2 - E_0^2}$ times RMS reading.

TABLE I

Point in the z-plane	Corresponding Value in the t-plane	Corresponding Value in the -plane
A	0	$-\ln \frac{VA}{V_j} + i \frac{\pi}{2}$
B	-k	$+\infty + i \left\{ \begin{matrix} c \\ \pi/2 \end{matrix} \right.$
C	-c	$-\ln \frac{VC}{V_j} + i 0$
D	-1	$0 + i 0$
E	f	$0 + i \beta$
F	1	$0 + i \pi$
G	g	$-\ln \frac{VG}{V_j} + i \pi$
H	k	$+\infty + i \left\{ \begin{matrix} \pi \\ \pi/2 \end{matrix} \right.$

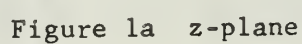


Figure 1 Transformations Planes

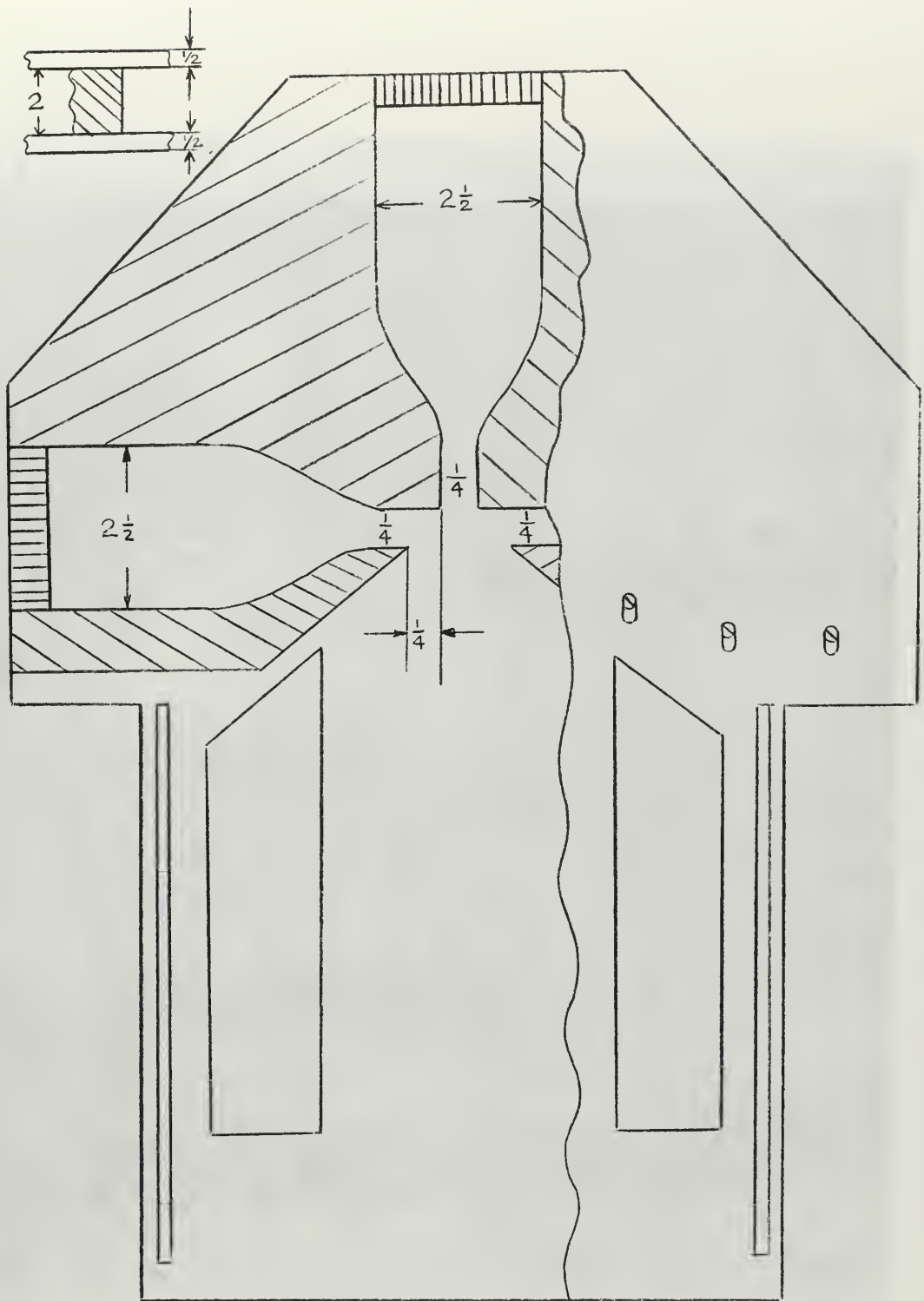


Figure 2 Jet Assembly

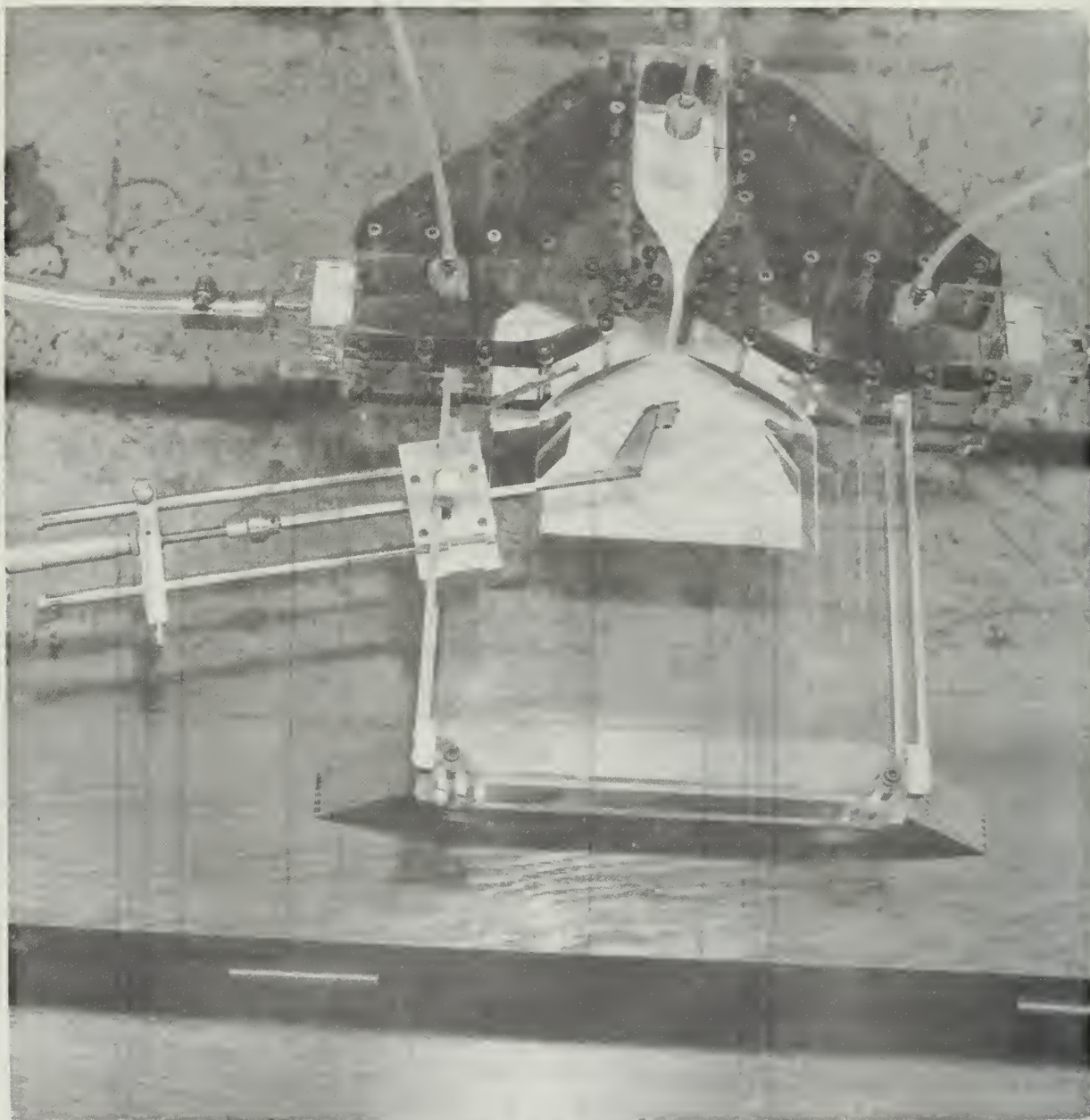


Figure 3 Jet Assembly



Figure 4 Calibration Equipment

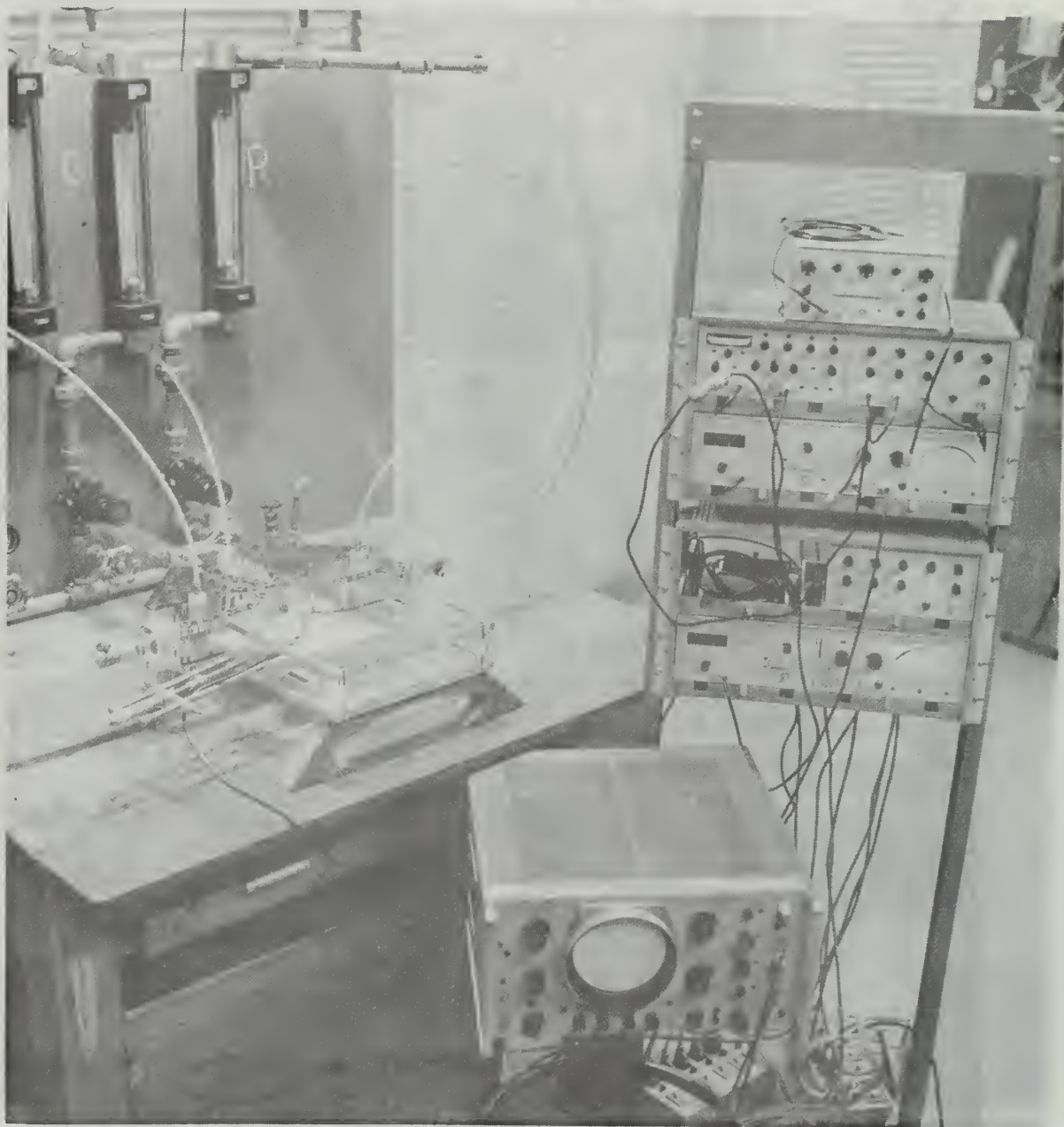


Figure 5 General Setup of Experiment

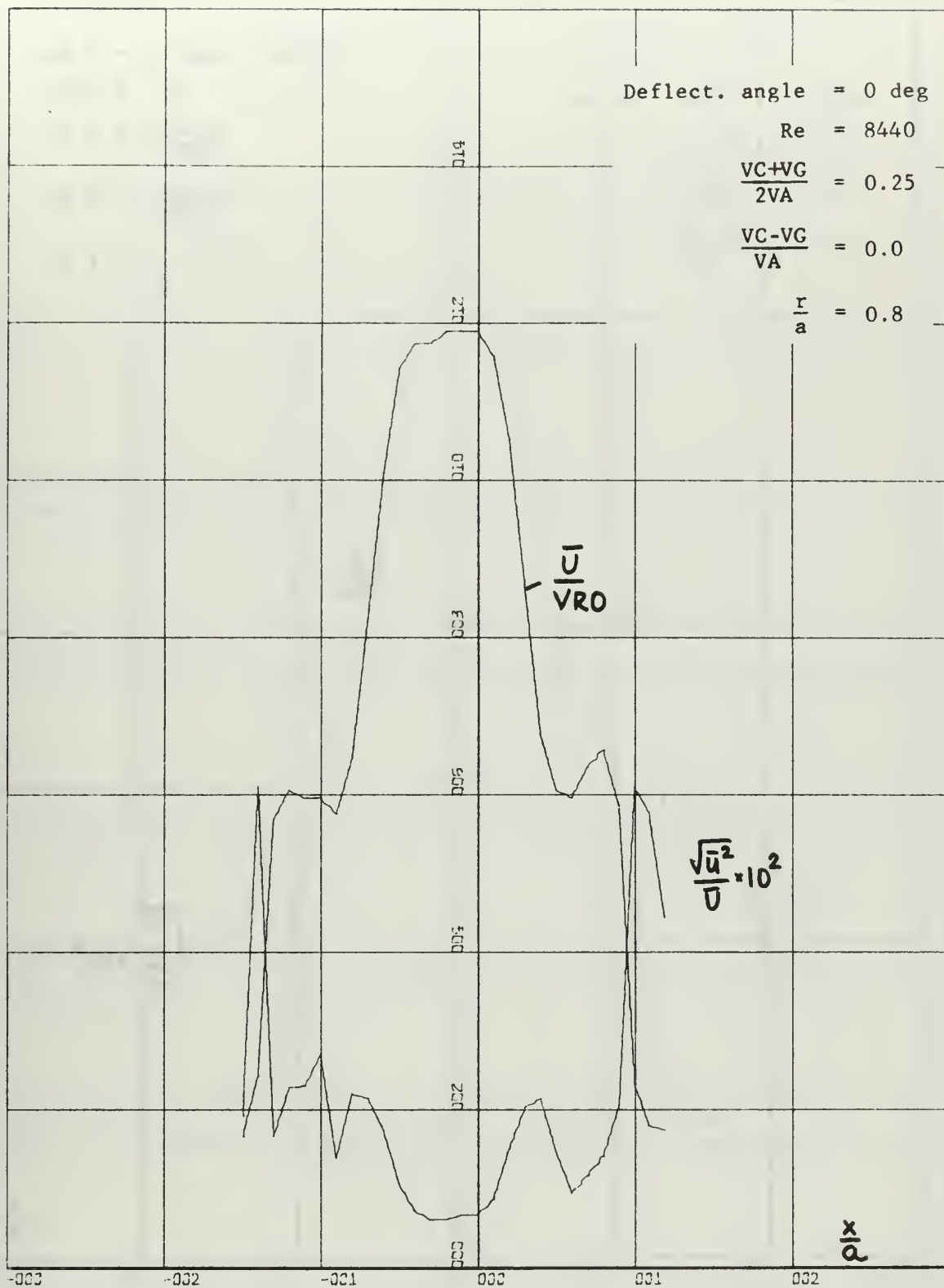


FIGURE 6 VELOCITY PROFILE AND TURBULENCE INTENSITY

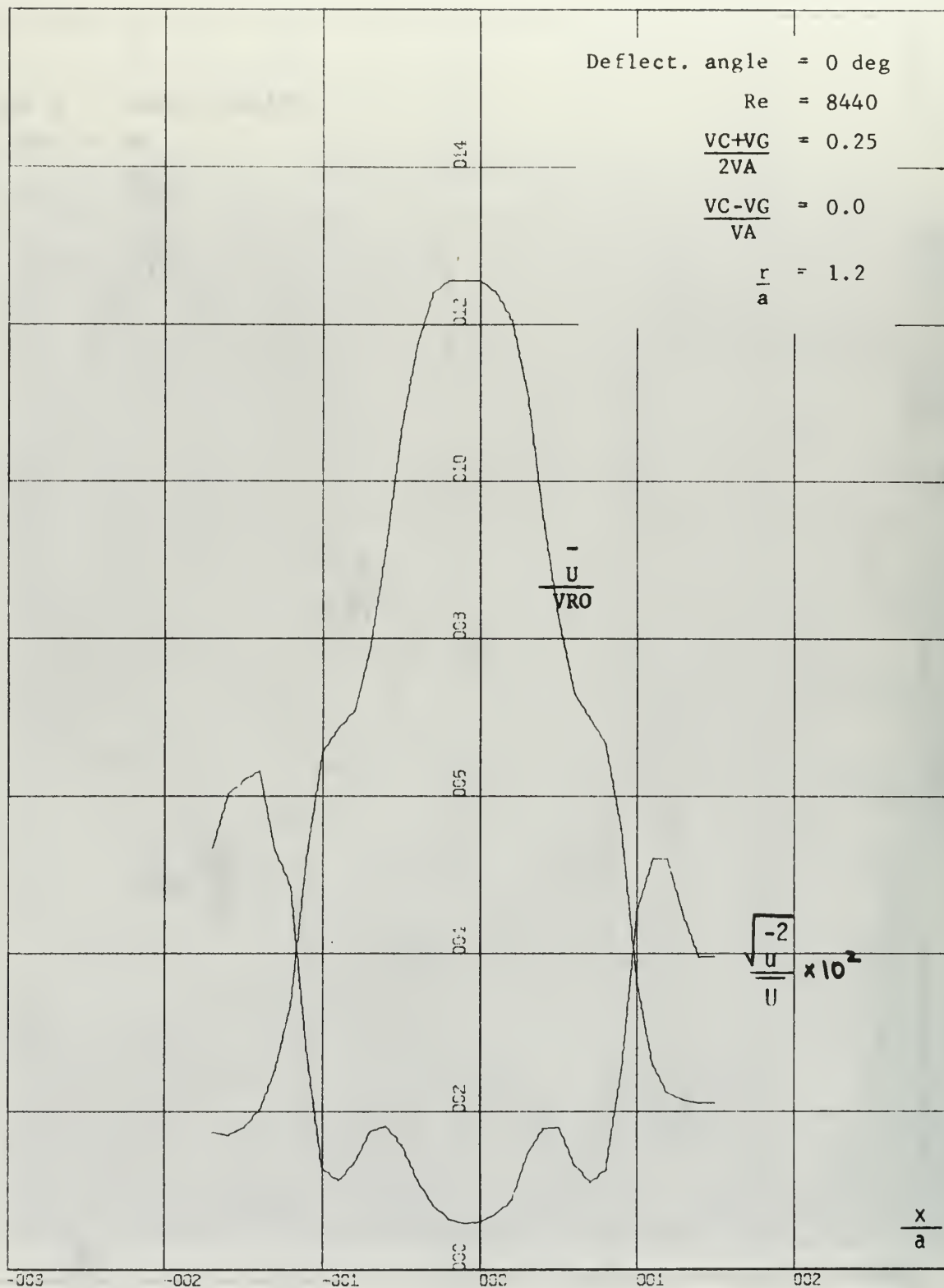


FIGURE 7 VELOCITY PROFILE AND TURBULENCE INTENSITY

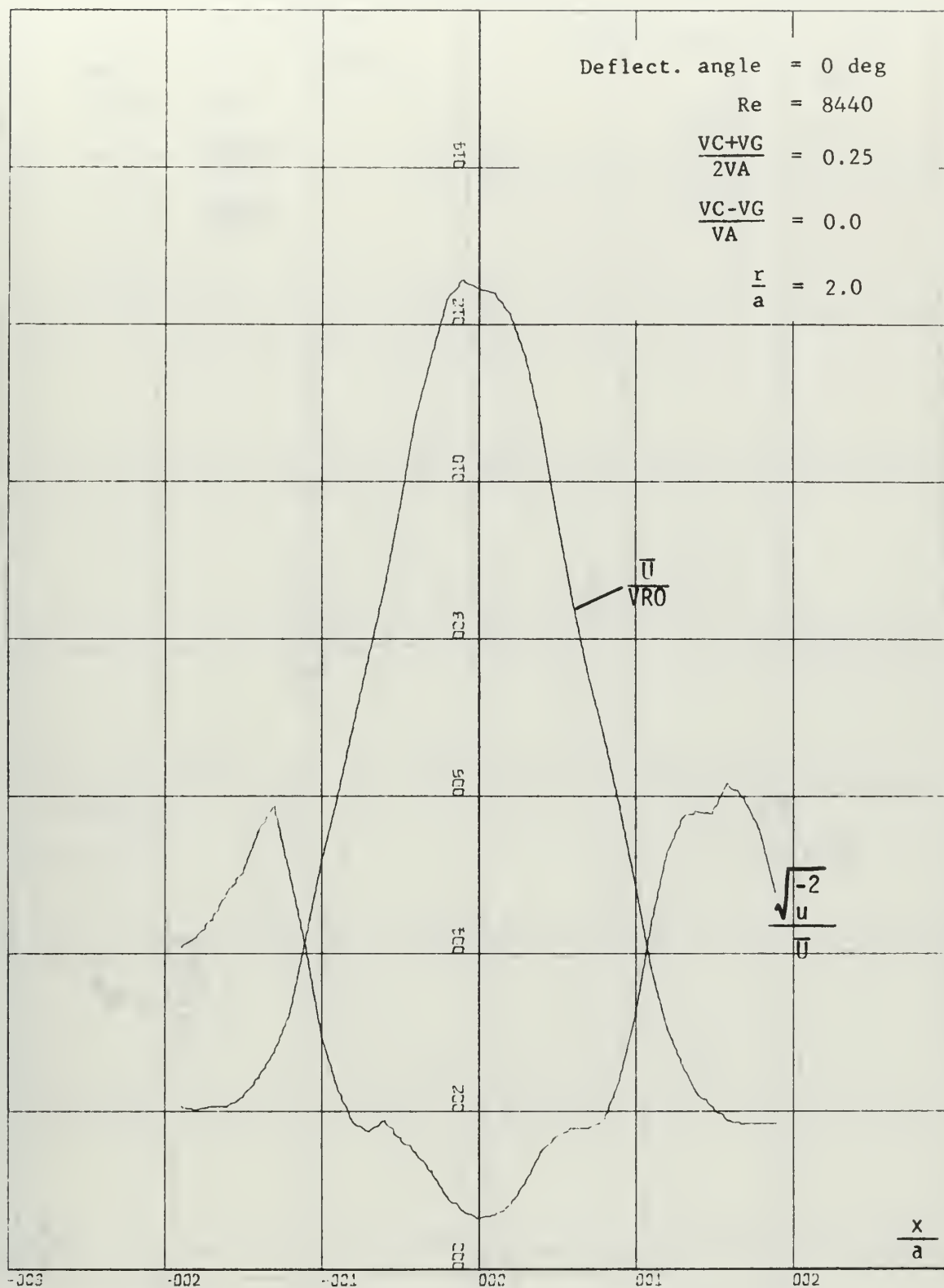


FIGURE 8 VELOCITY PROFILE AND TURBULENCE INTENSITY

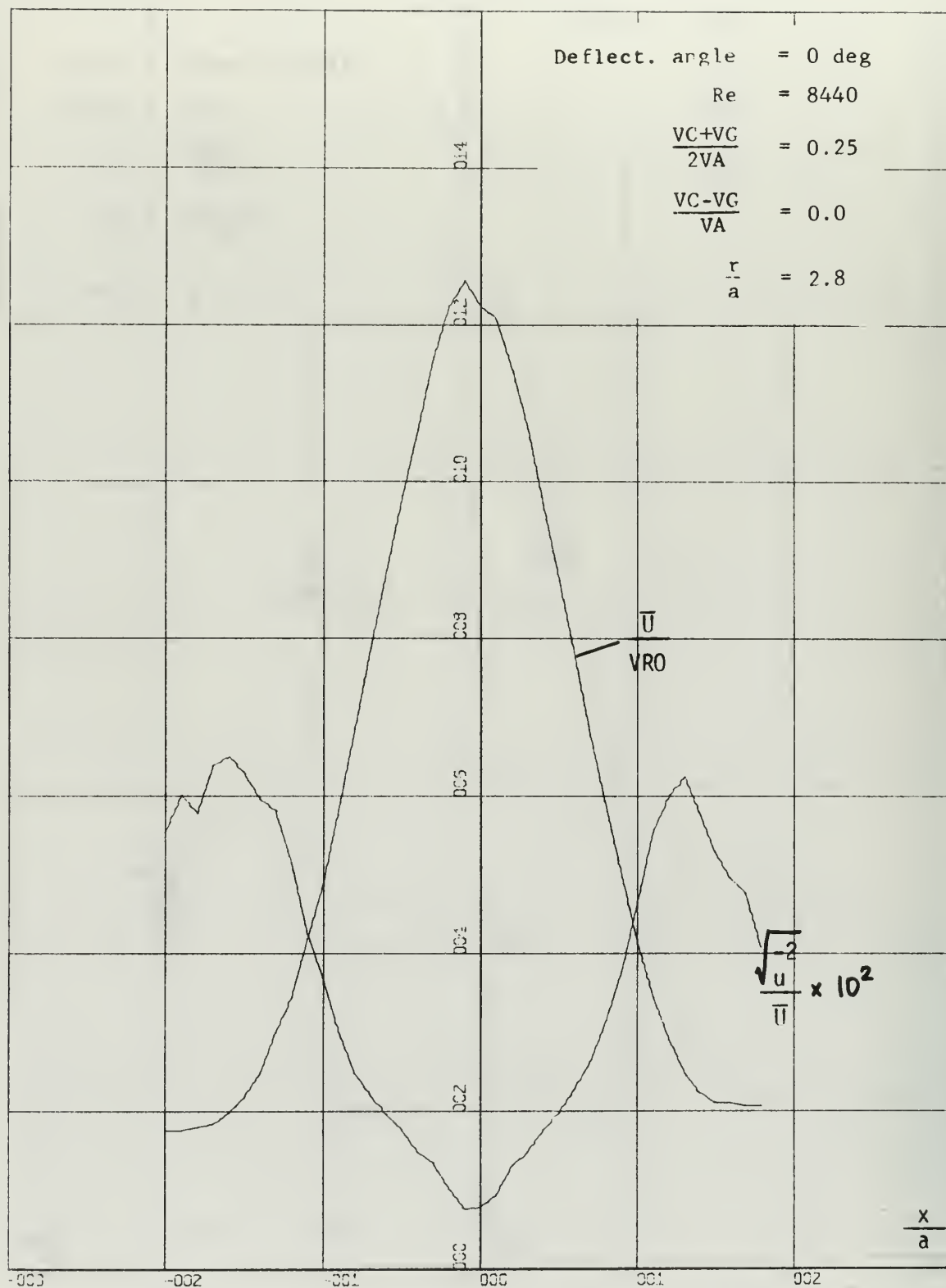


FIGURE 9 VELOCITY PROFILE AND TURBULENCE INTENSITY

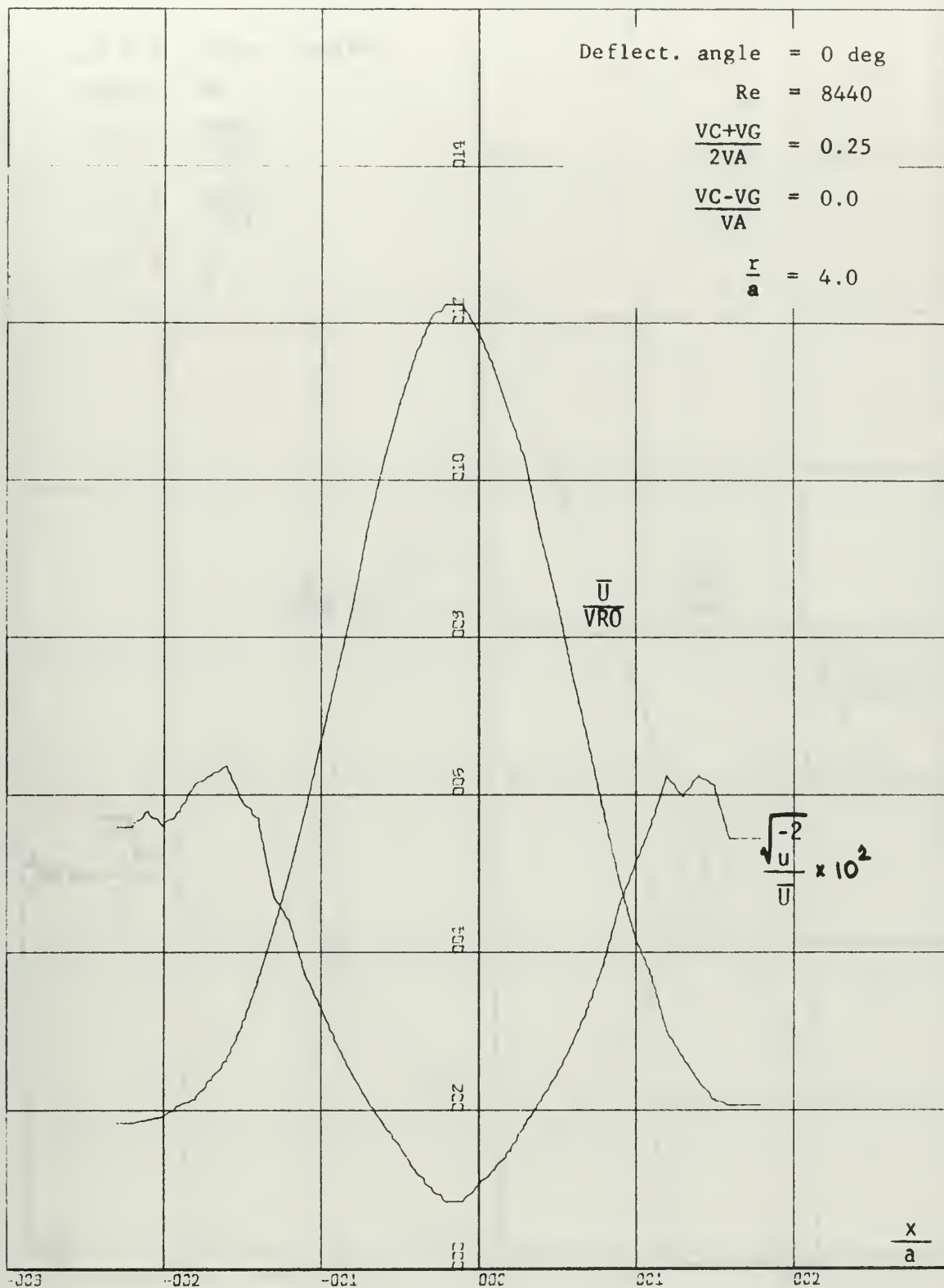


FIGURE 10 VELOCITY PROFILE AND TURBULENCE INTENSITY

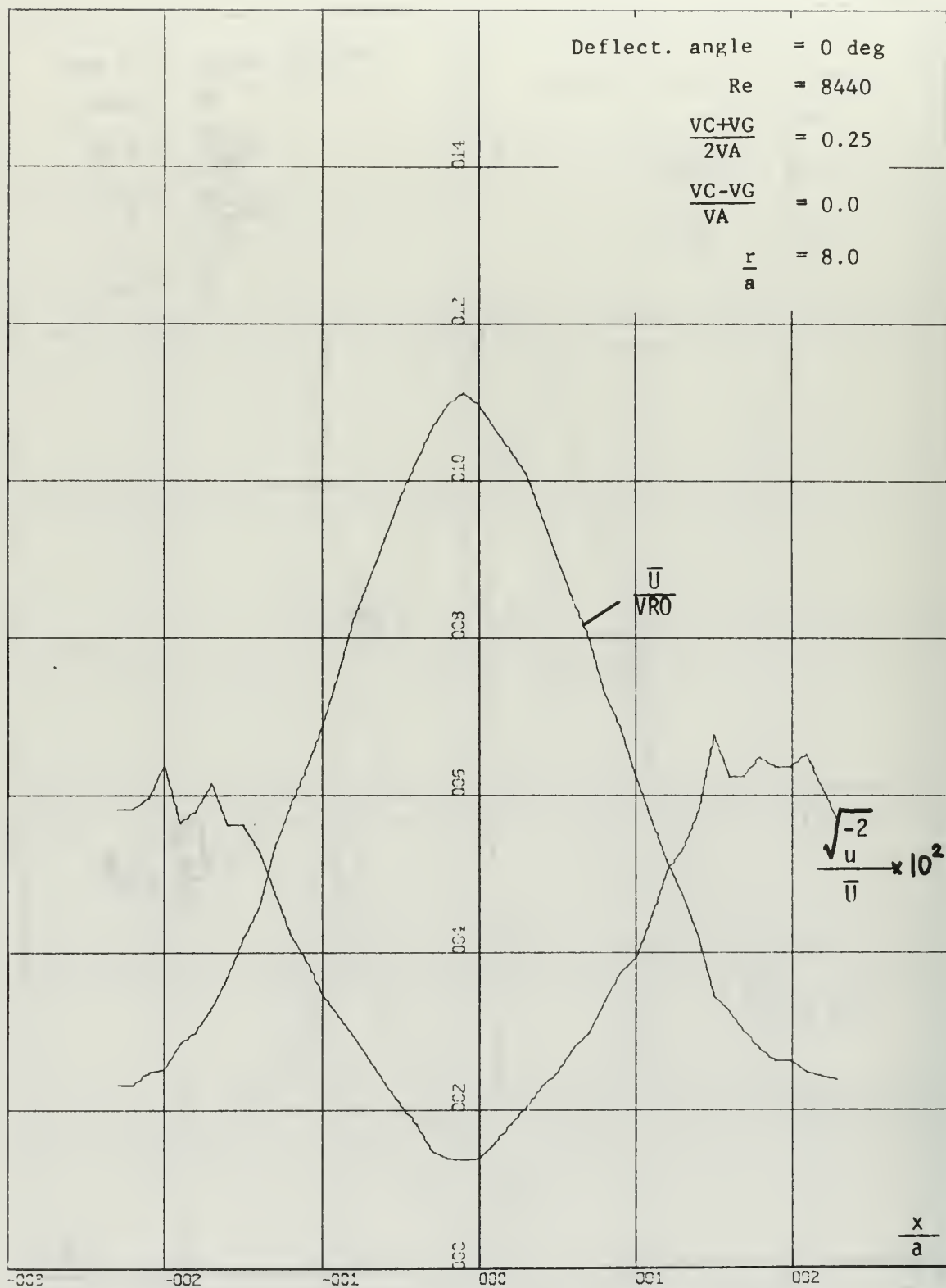


FIGURE 11 VELOCITY PROFILE AND TURBULENCE INTENSITY

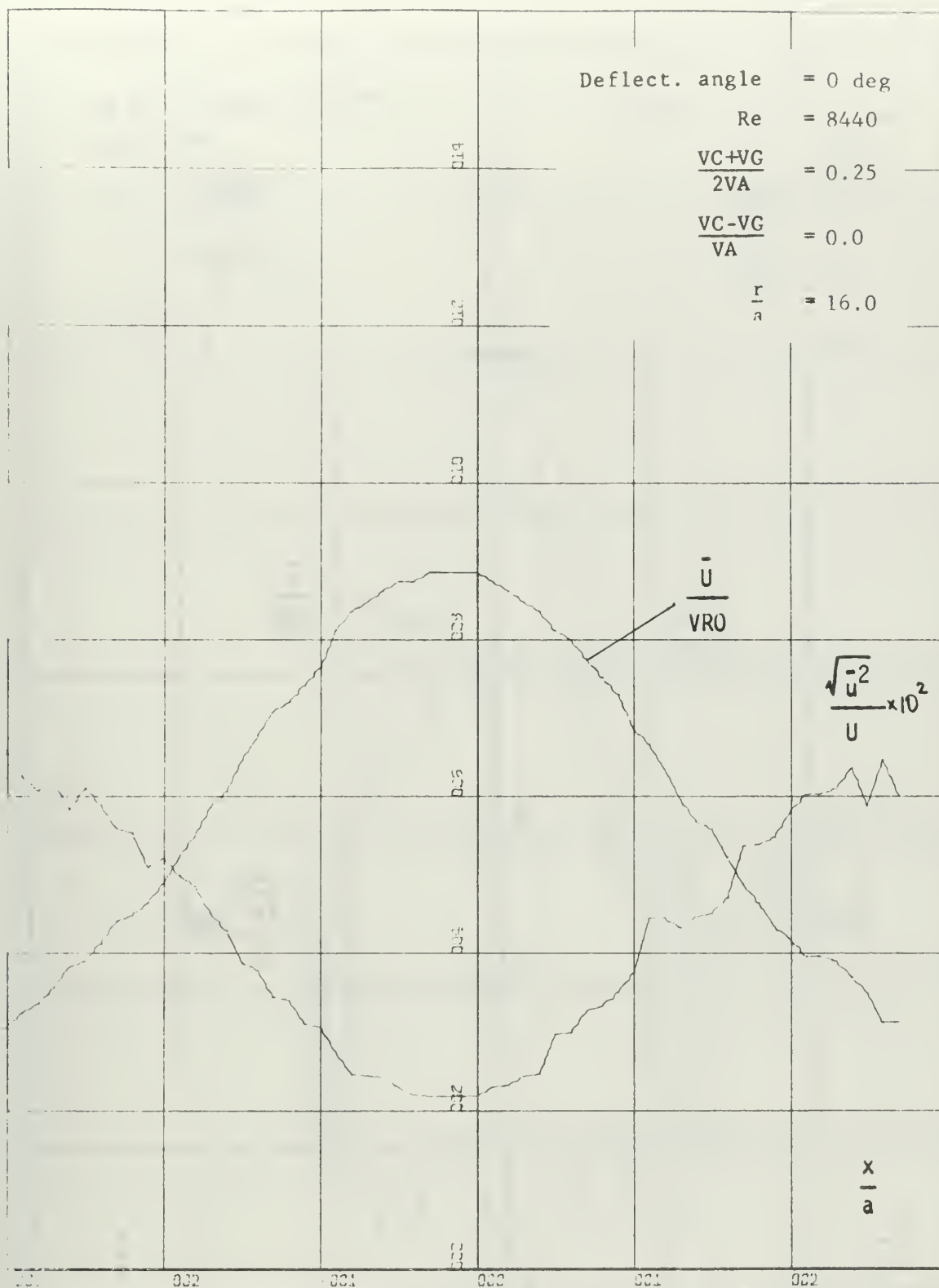


FIGURE 12 VELOCITY PROFILE AND TURBULENCE INTENSITY

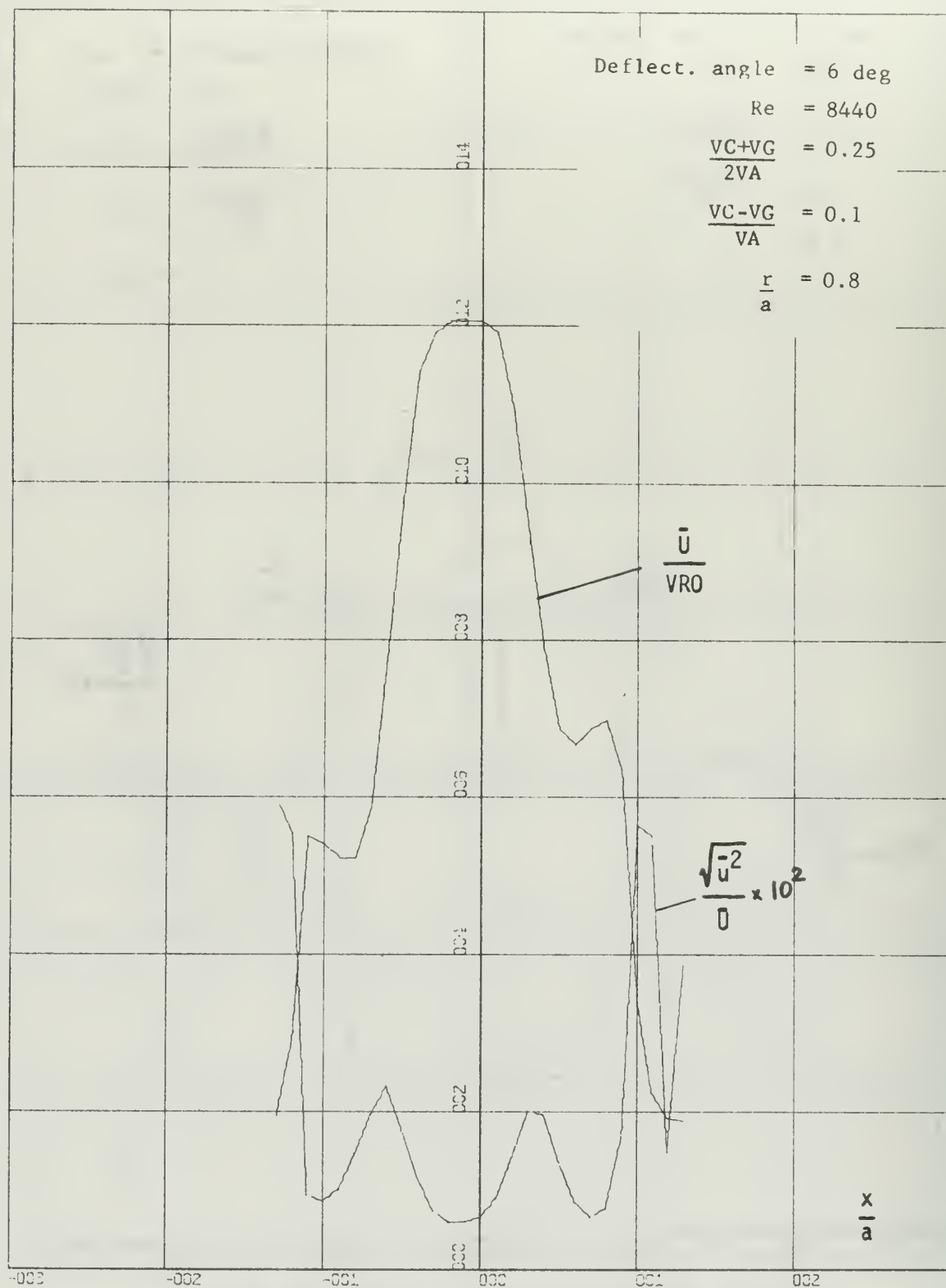


FIGURE 13 VELOCITY PROFILE AND TURBULENCE INTENSITY

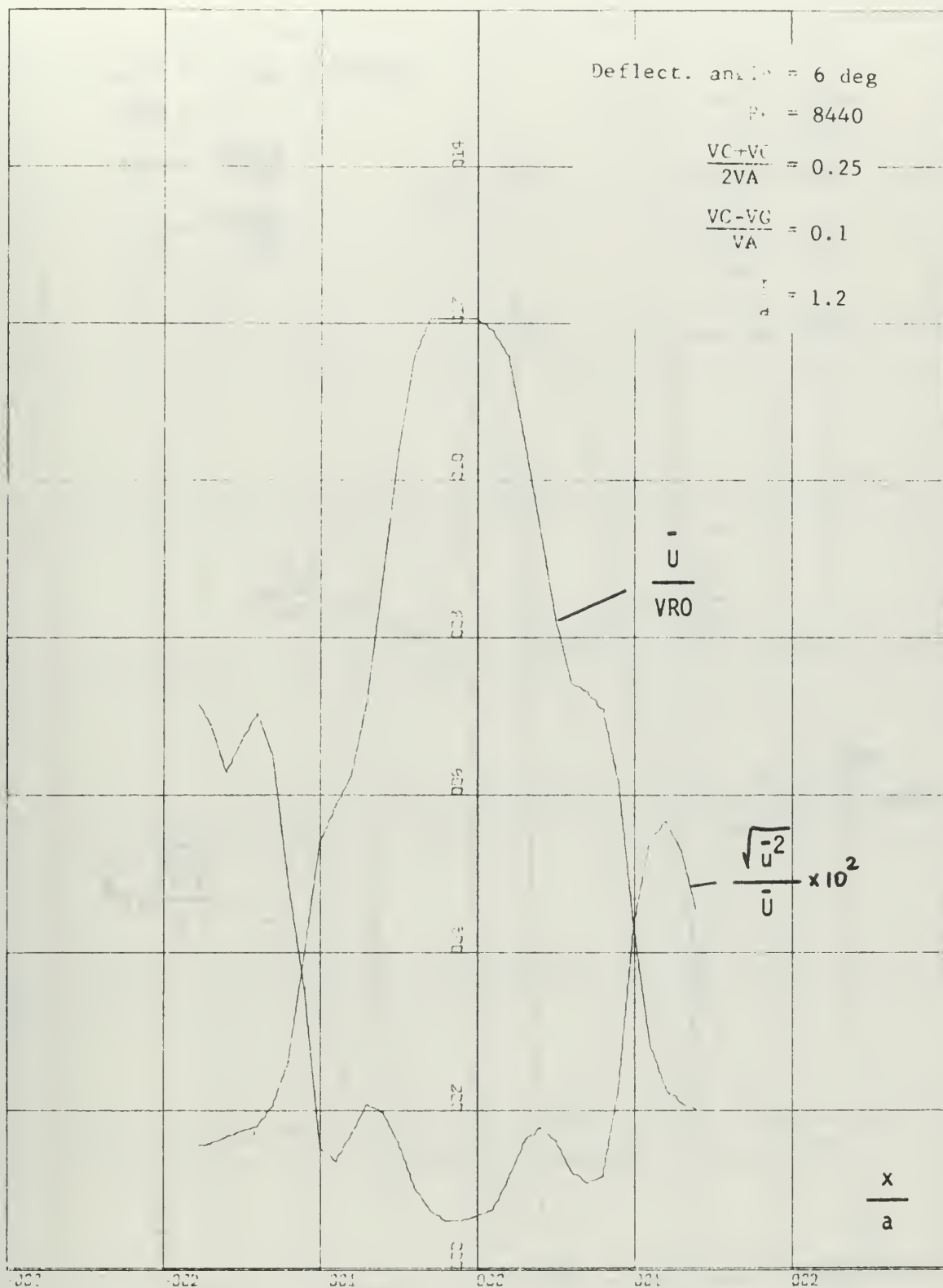


FIGURE 14 VELOCITY PROFILE AND TURBULENCE INTENSITY

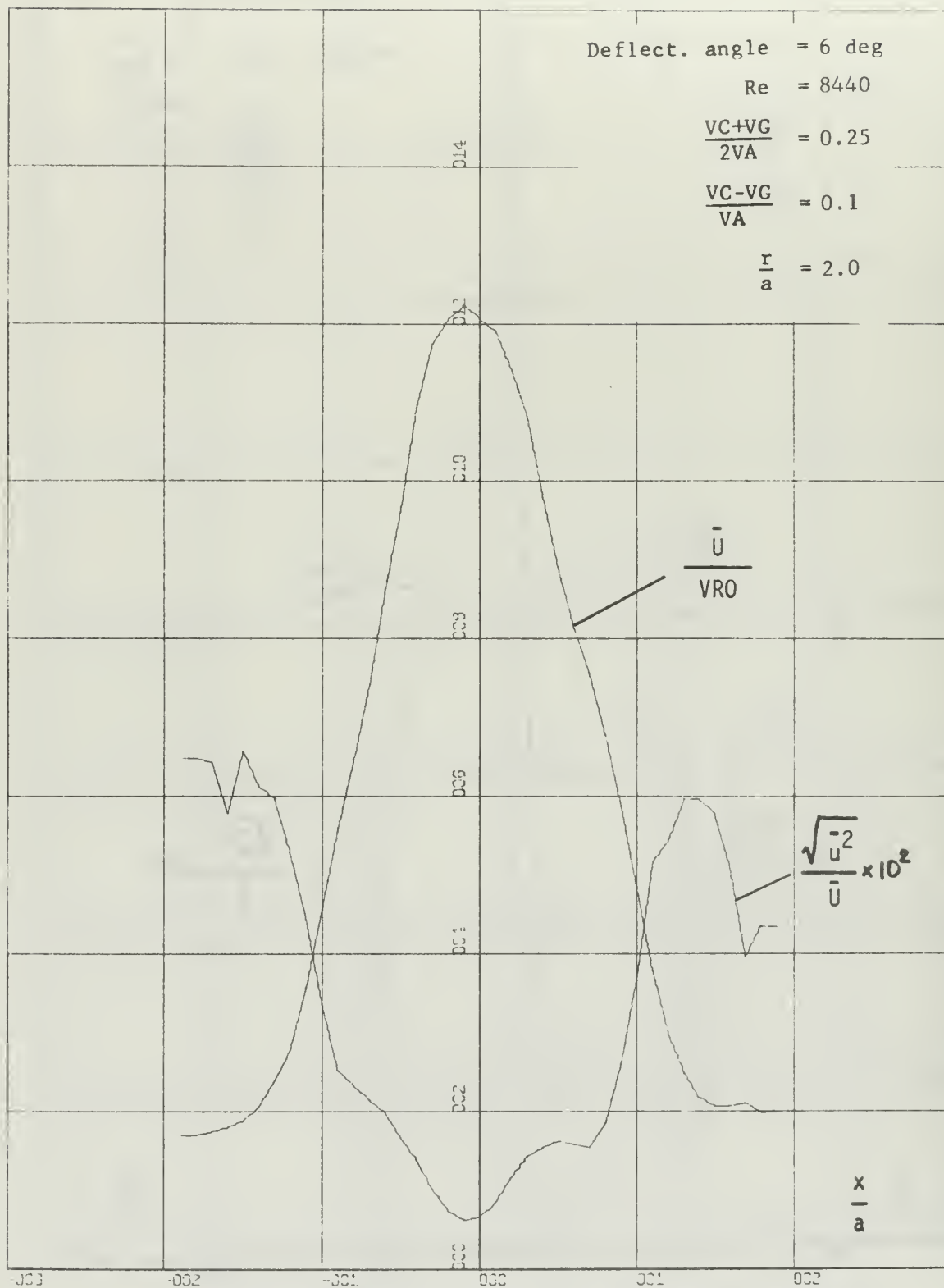


FIGURE 15 VELOCITY PROFILE AND TURBULENCE INTENSITY

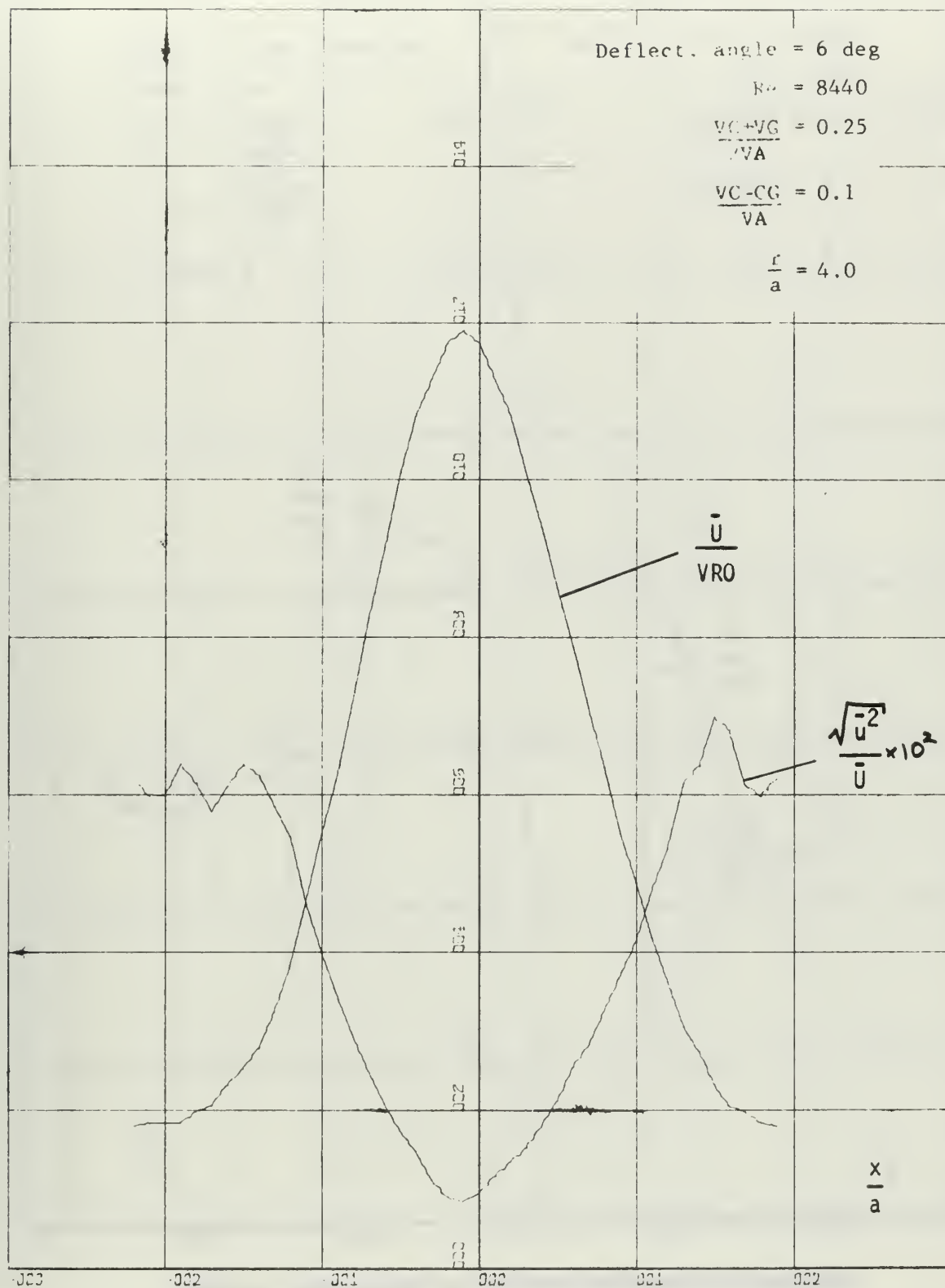


FIGURE 16 VELOCITY PROFILE AND TURBULENCE INTENSITY

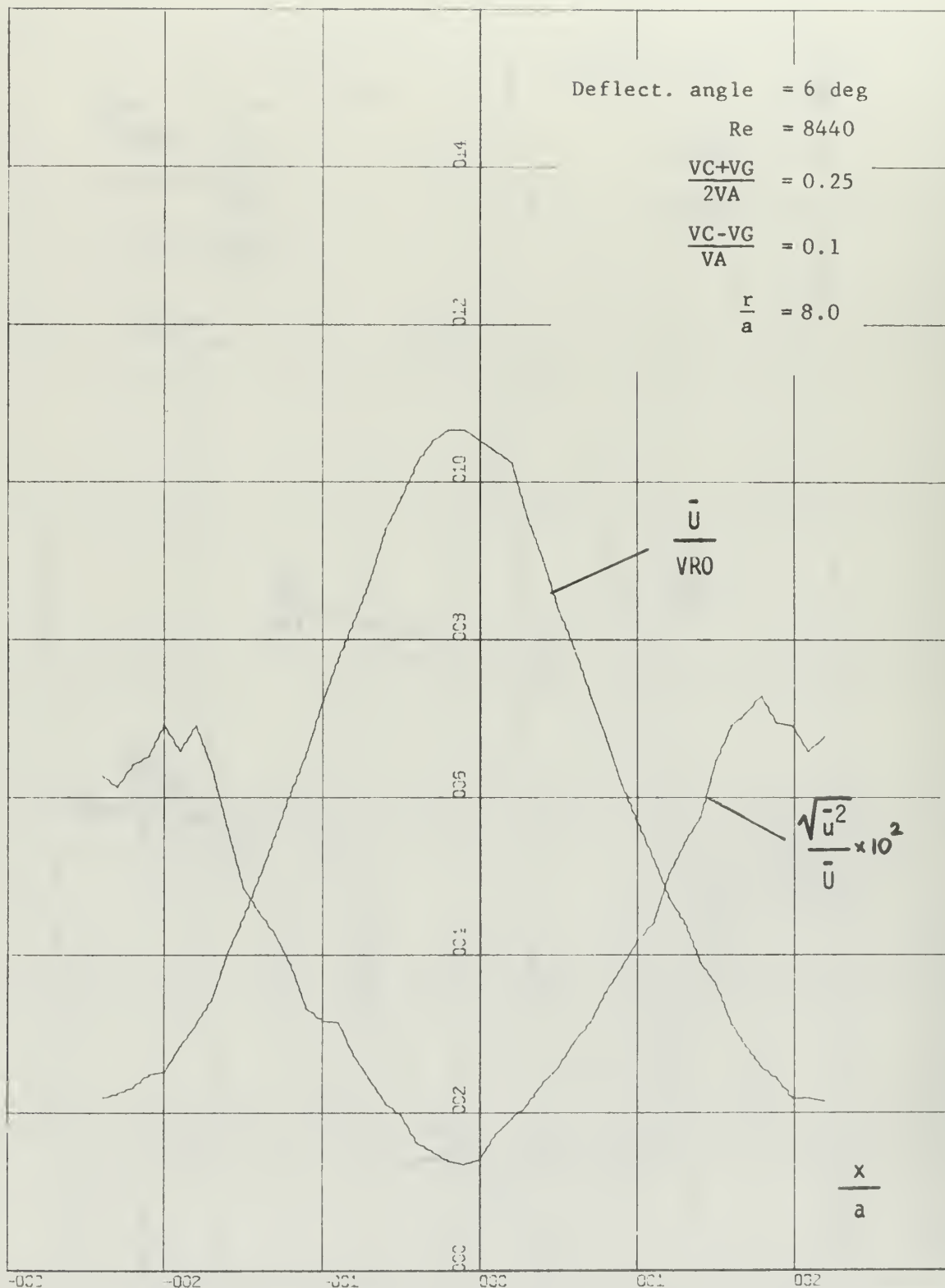


FIGURE 17 VELOCITY PROFILE AND TURBULENCE INTENSITY

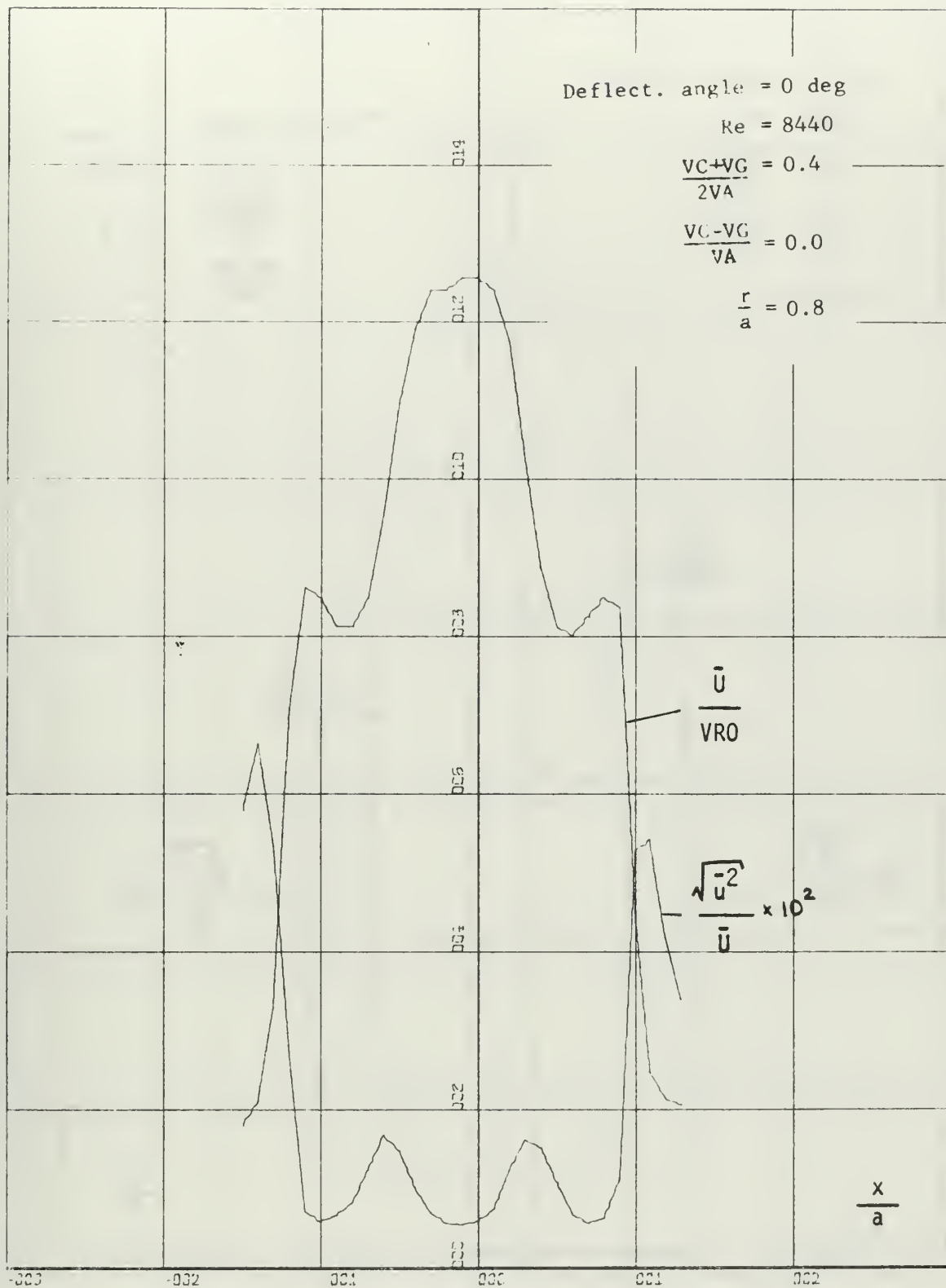


FIGURE 18 VELOCITY PROFILE AND TURBULENCE INTENSITY

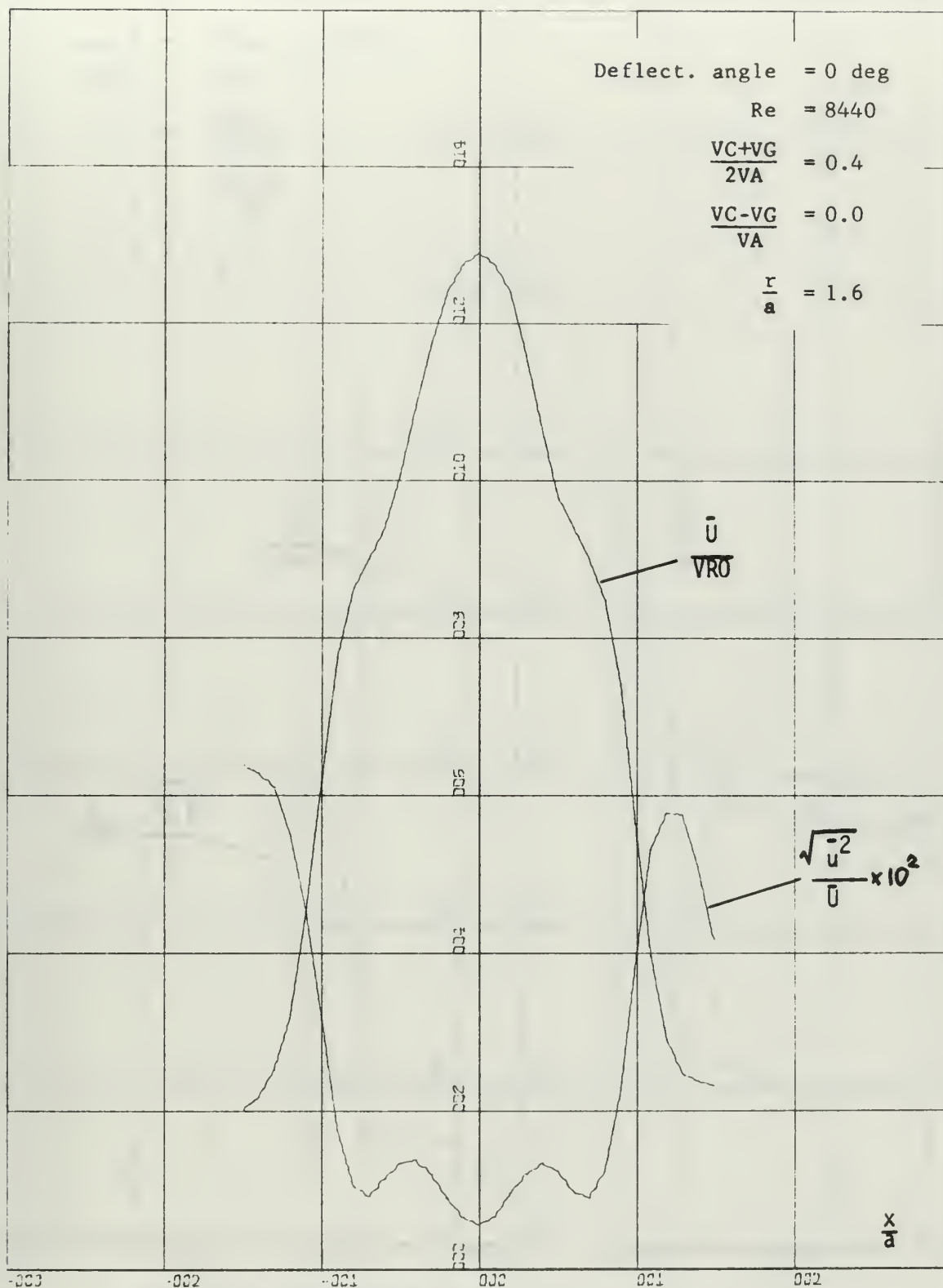


FIGURE 20 VELOCITY PROFILE AND TURBULENCE INTENSITY

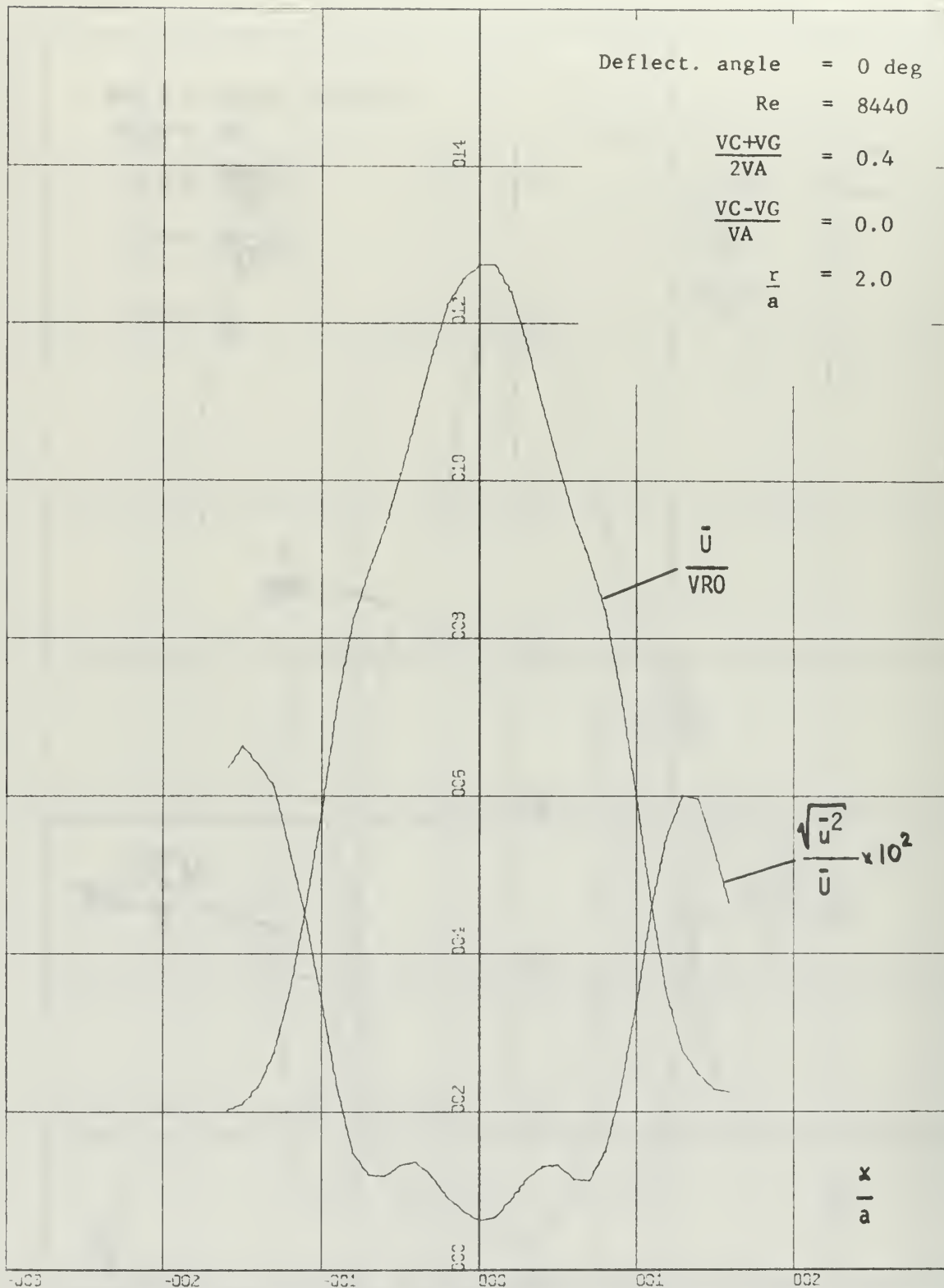


FIGURE 21 VELOCITY PROFILE AND TURBULENCE INTENSITY

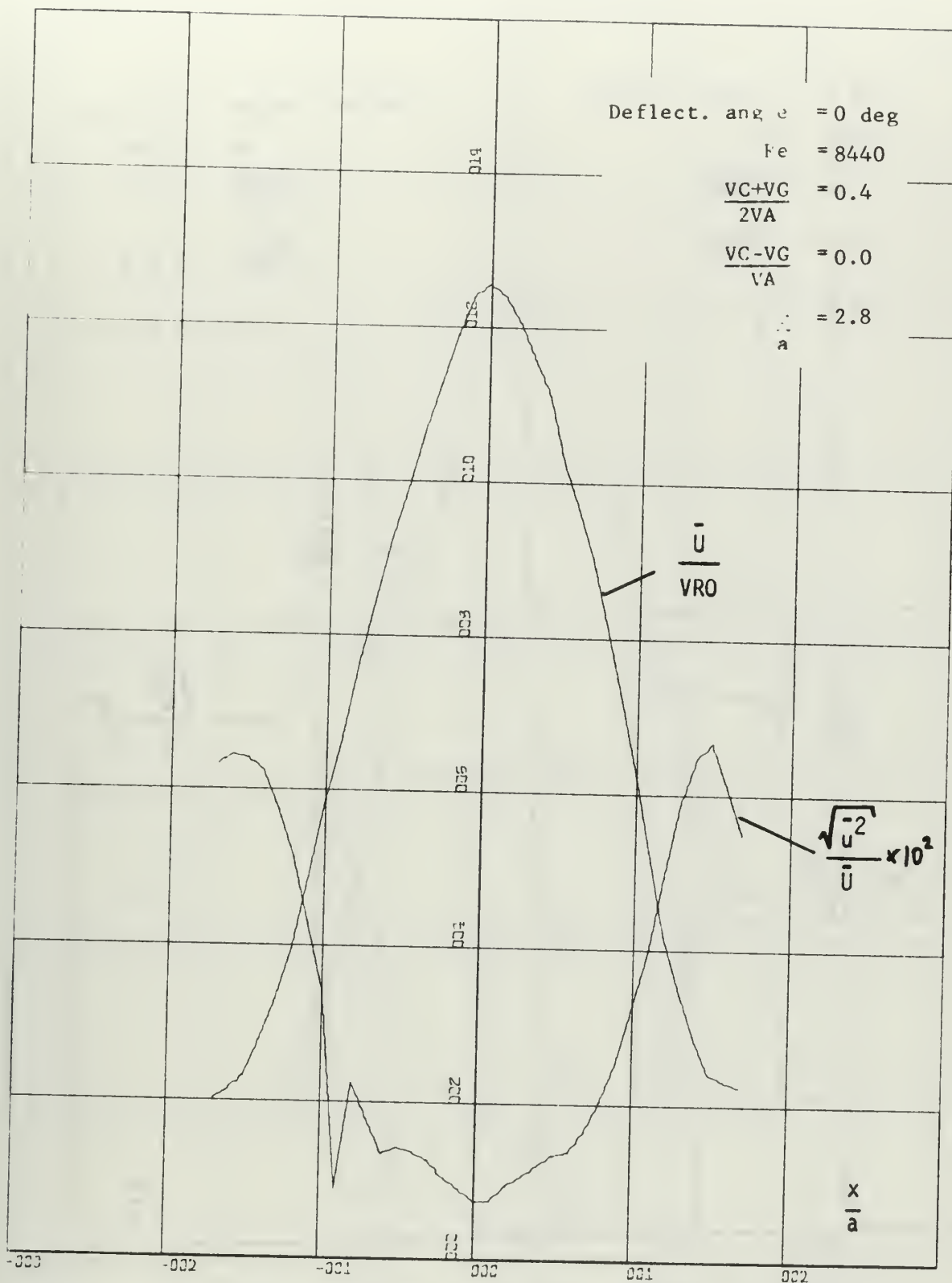


FIGURE 22 VELOCITY PROFILE AND TURBULENCE INTENSITY

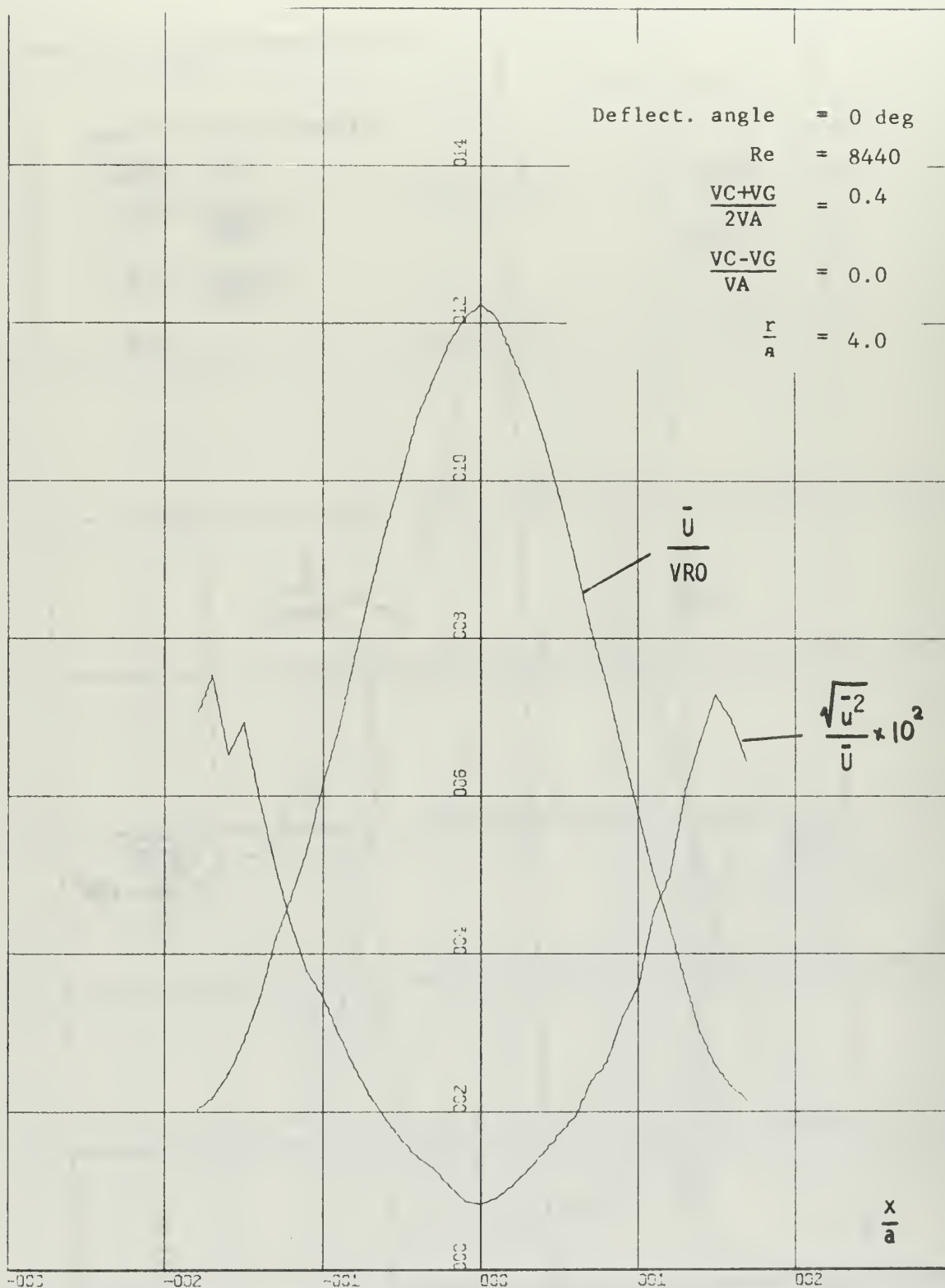


FIGURE 23 VELOCITY PROFILE AND TURBULENCE INTENSITY

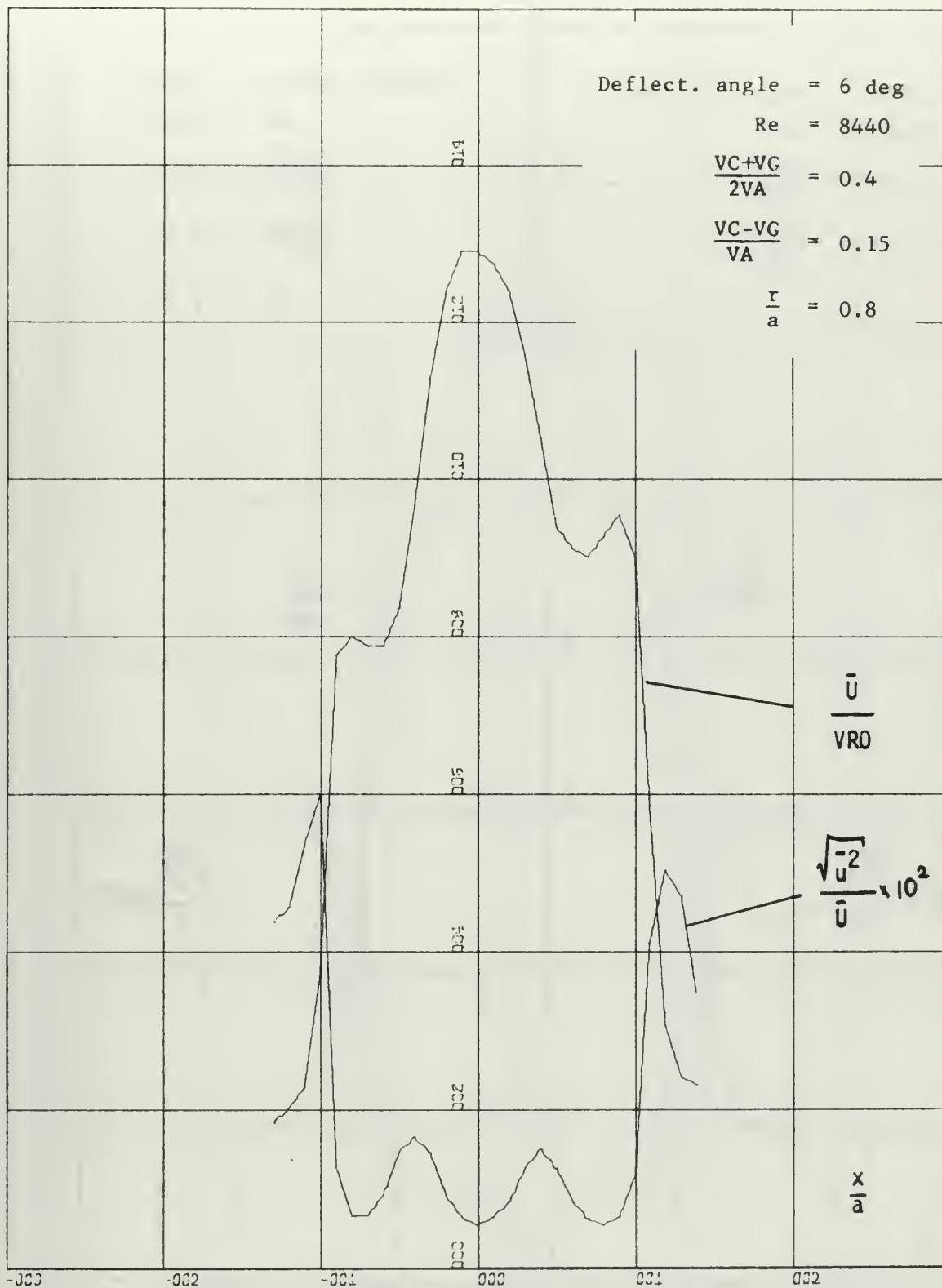


FIGURE 24 VELOCITY PROFILE AND TURBULENCE INTENSITY

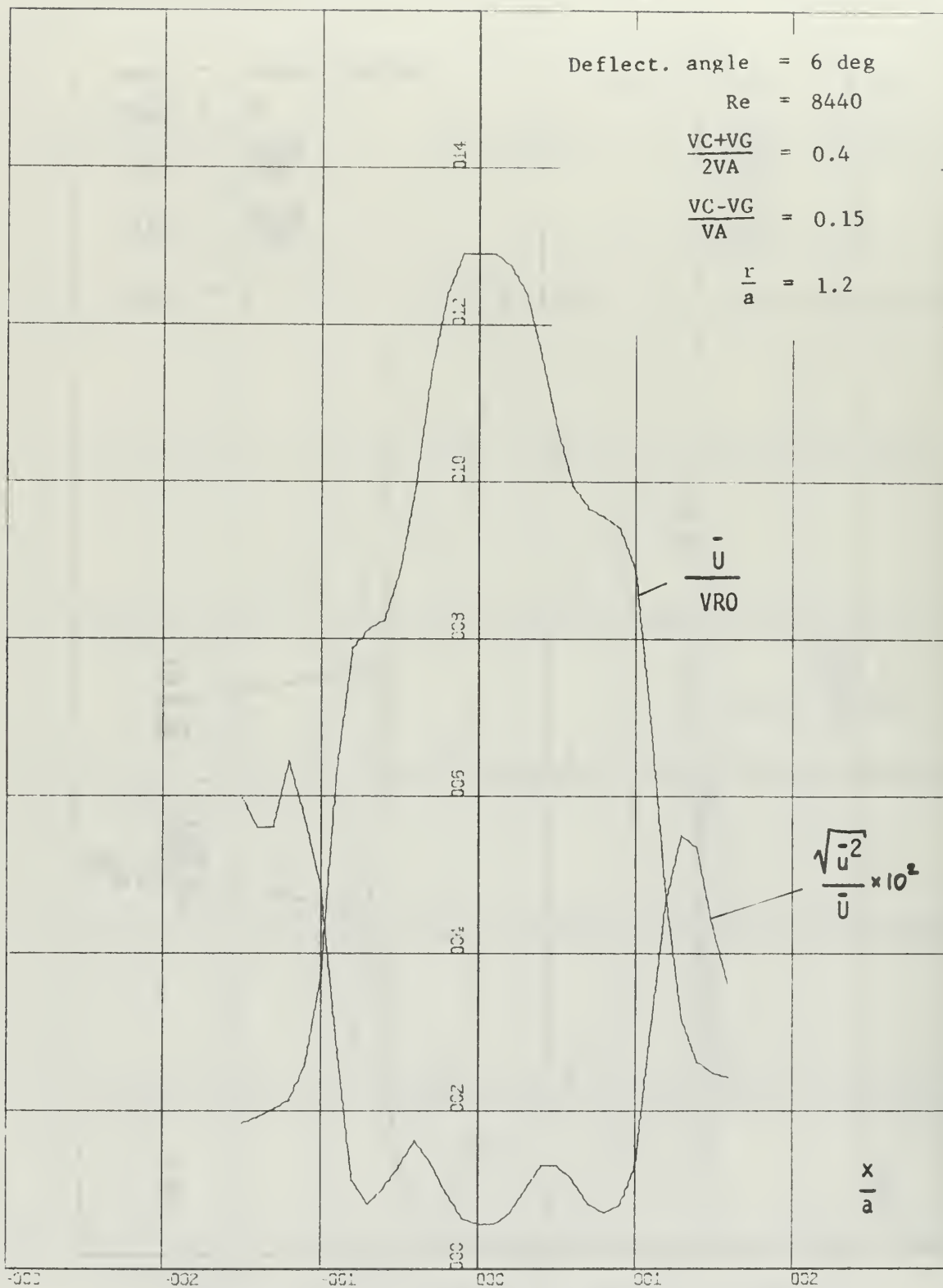


FIGURE 25 VELOCITY PROFILE AND TURBULENCE INTENSITY

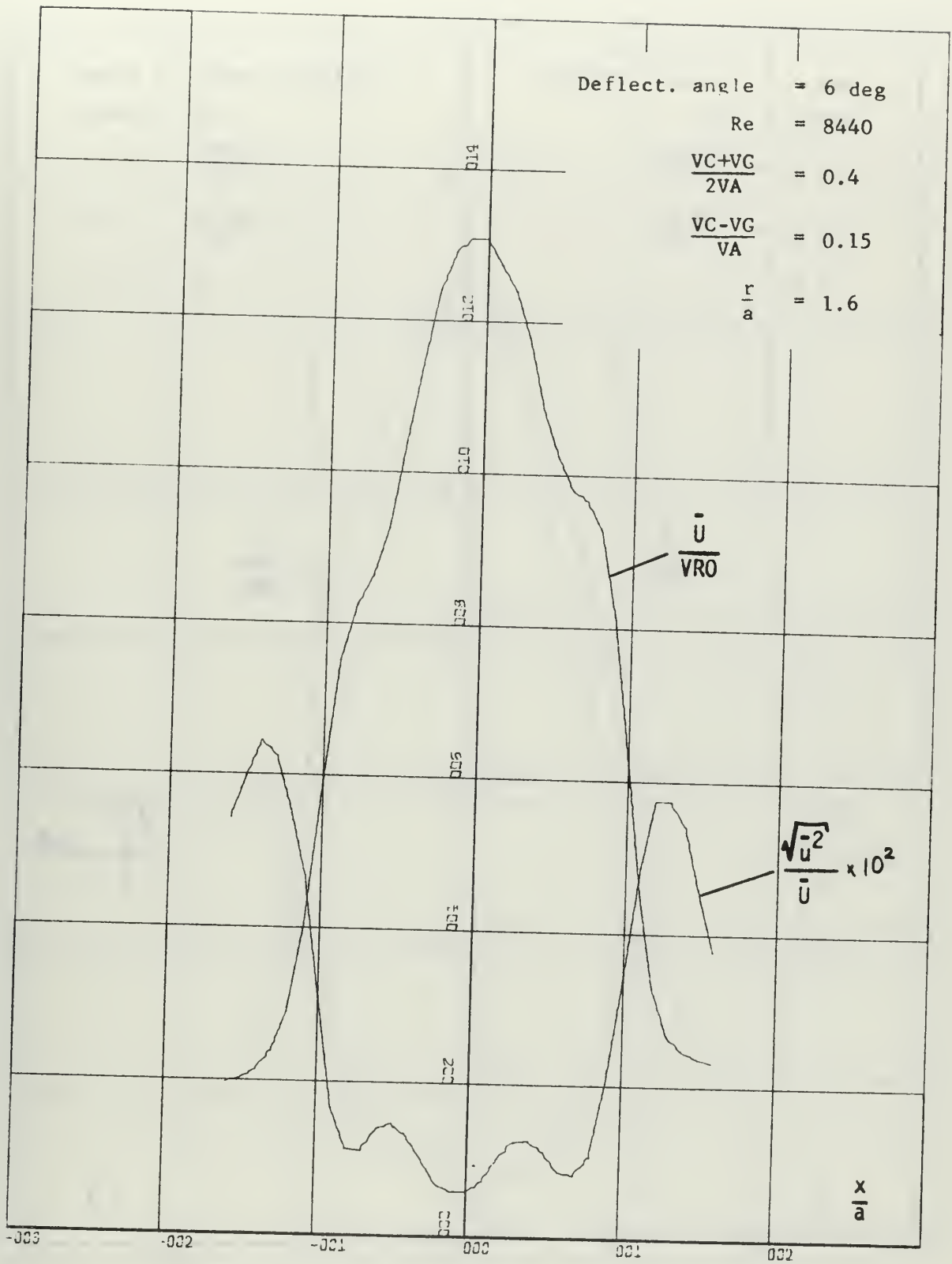


FIGURE 26 VELOCITY PROFILE AND TURBULENCE INTENSITY

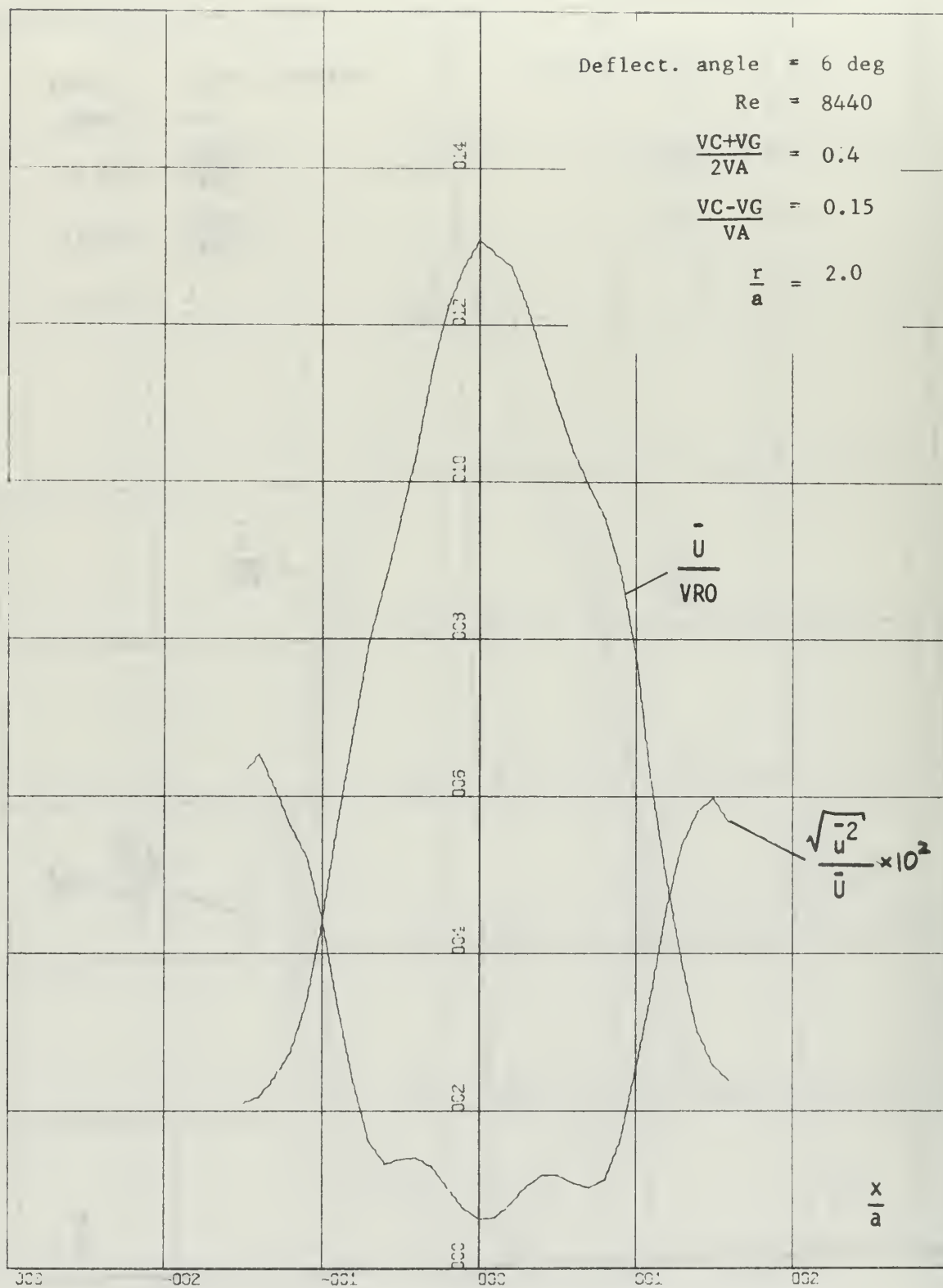


FIGURE 27 VELOCITY PROFILE AND TURBULENCE INTENSITY

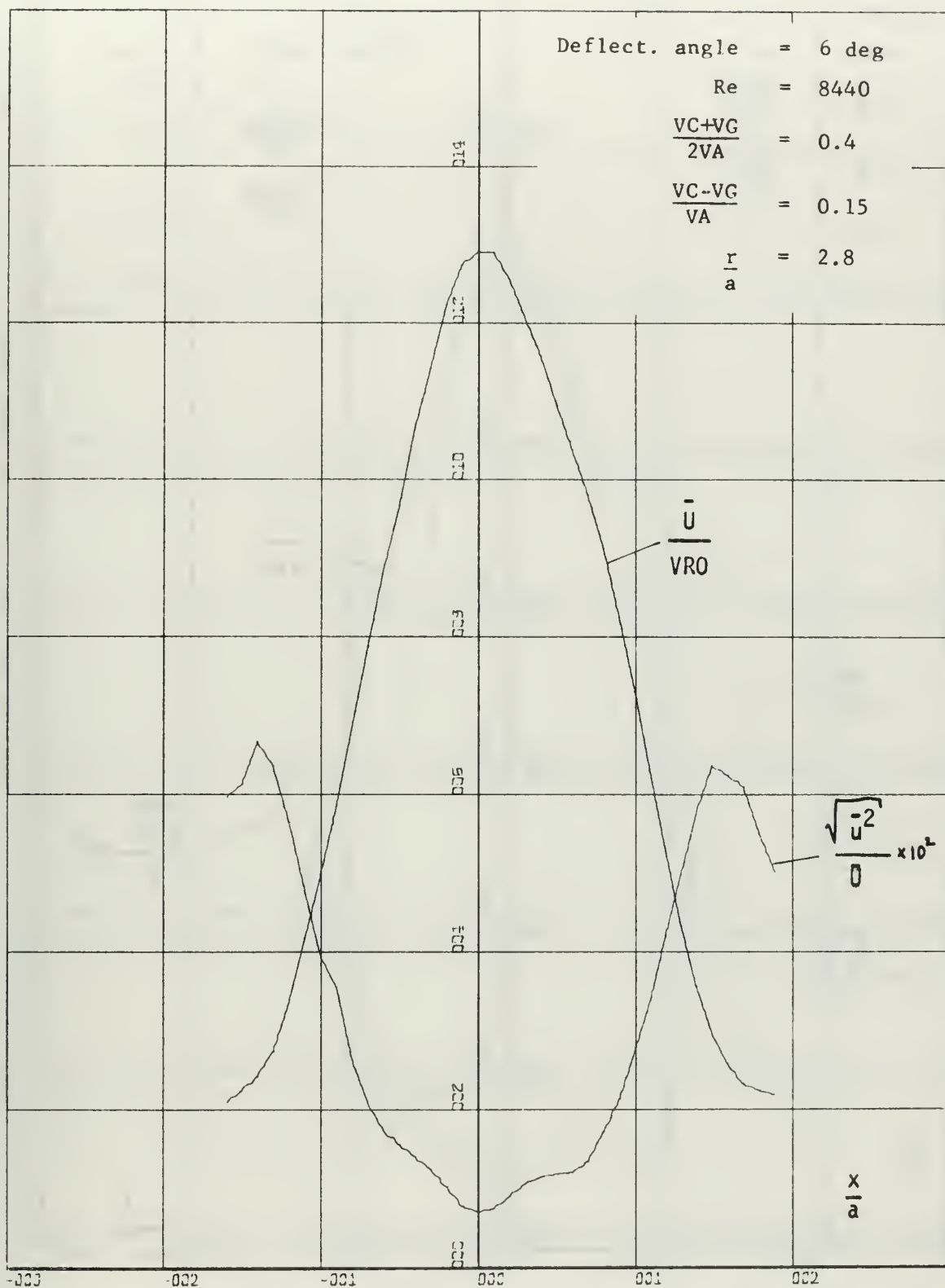


FIGURE 28 VELOCITY PROFILE AND TURBULENCE INTENSITY

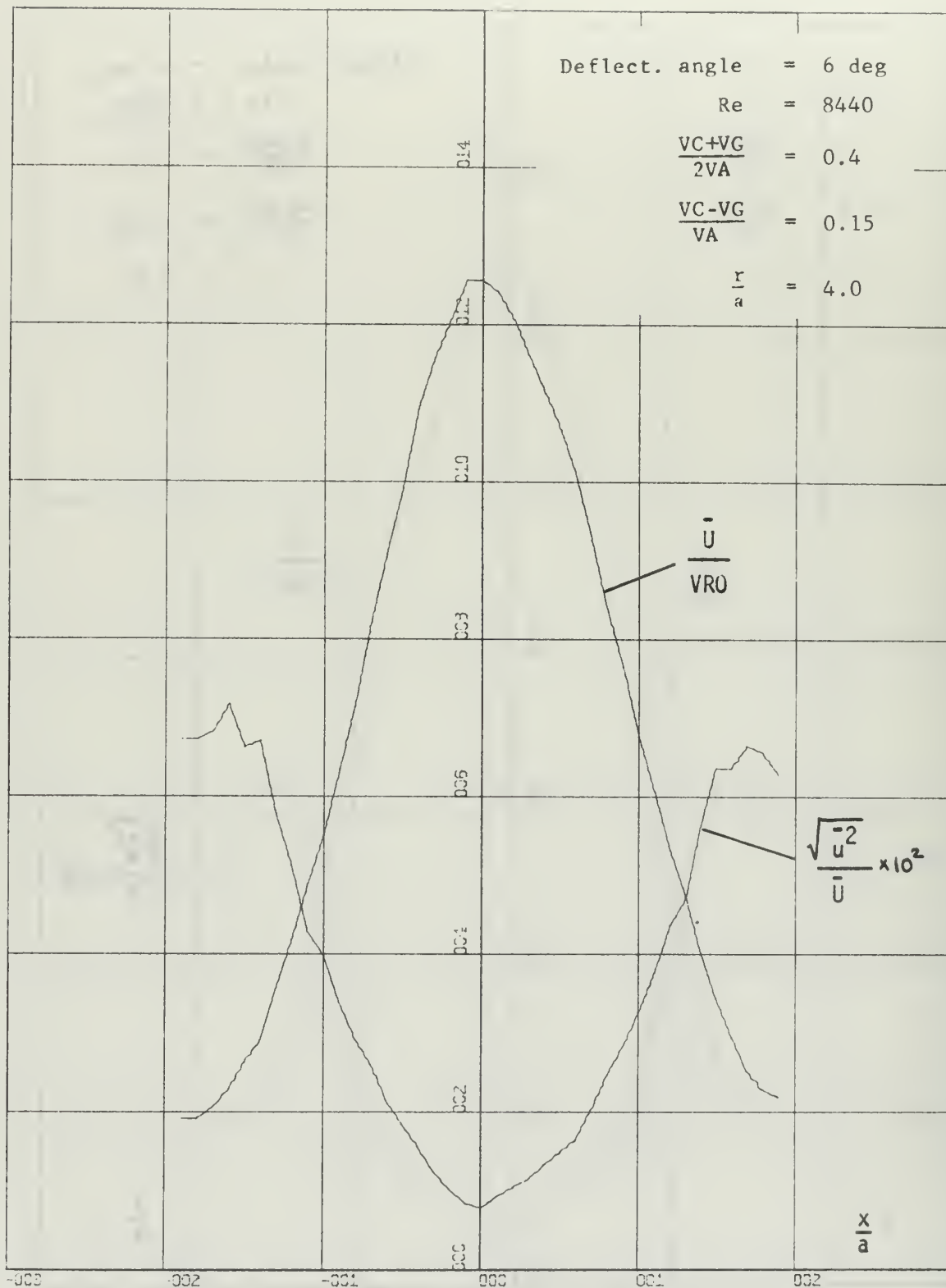


FIGURE 29 VELOCITY PROFILE AND TURBULENCE INTENSITY

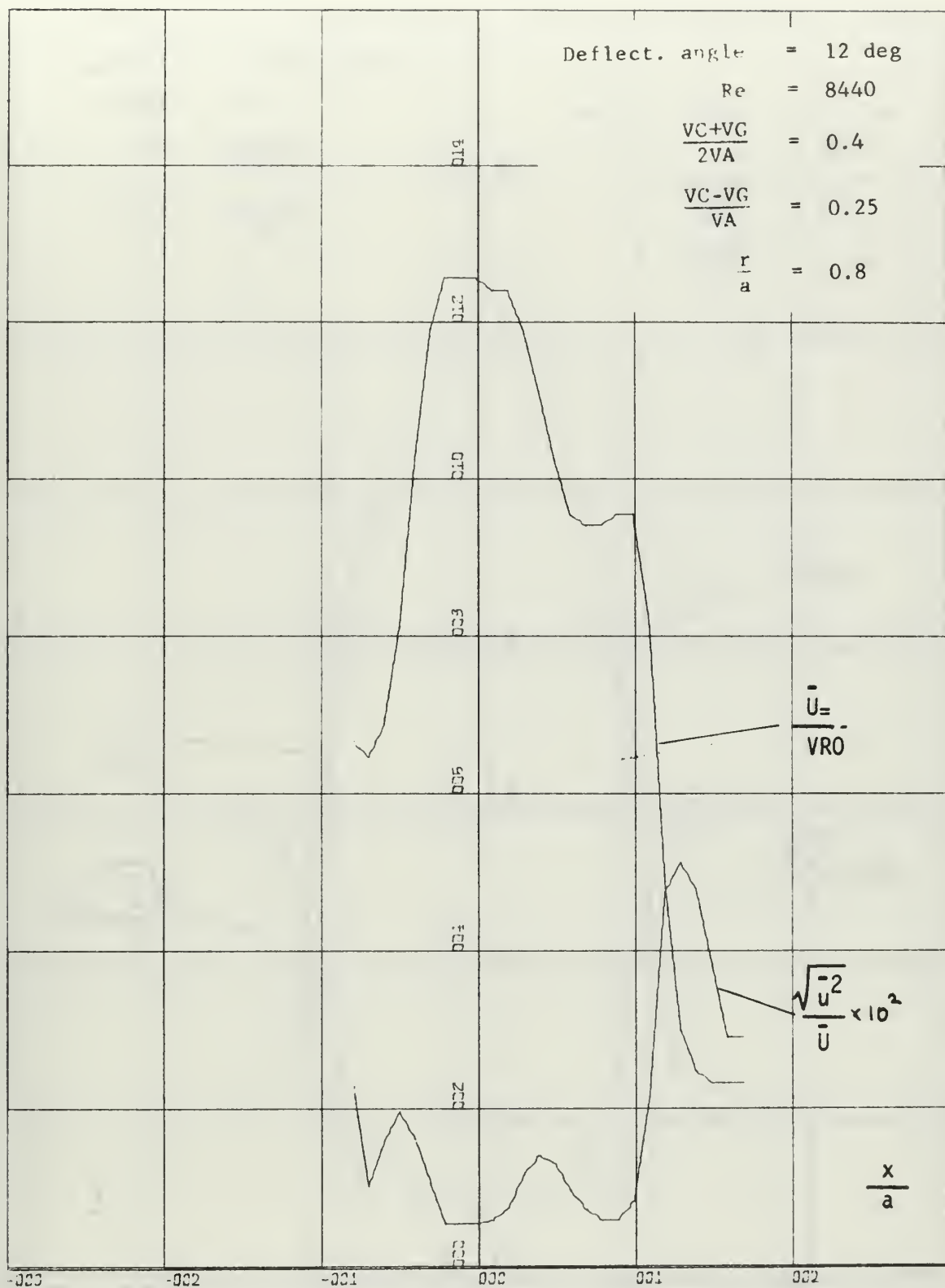


FIGURE 30 VELOCITY PROFILE AND TURBULENCE INTENSITY

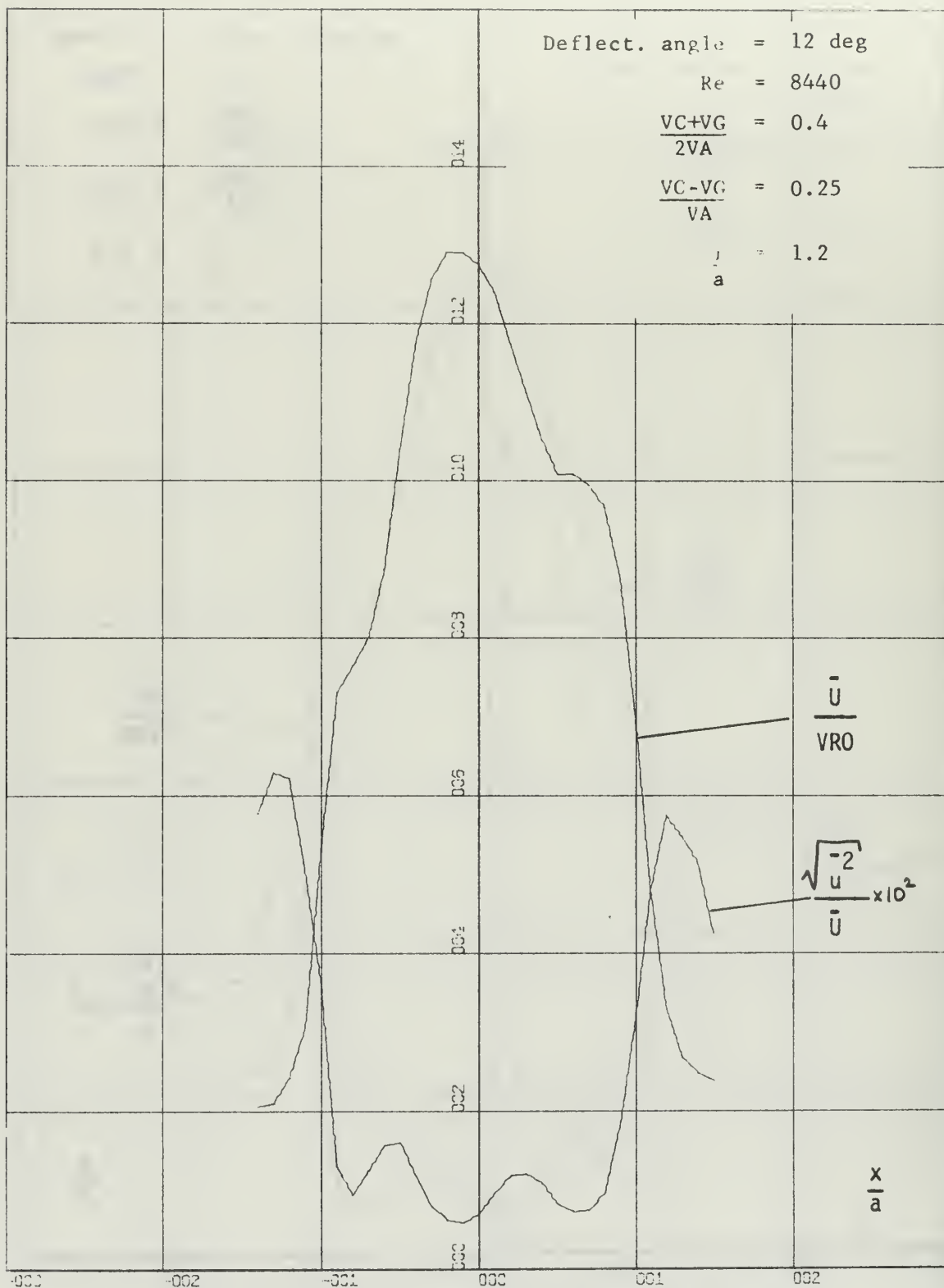


FIGURE 31 VELOCITY PROFILE AND TURBULENCE INTENSITY

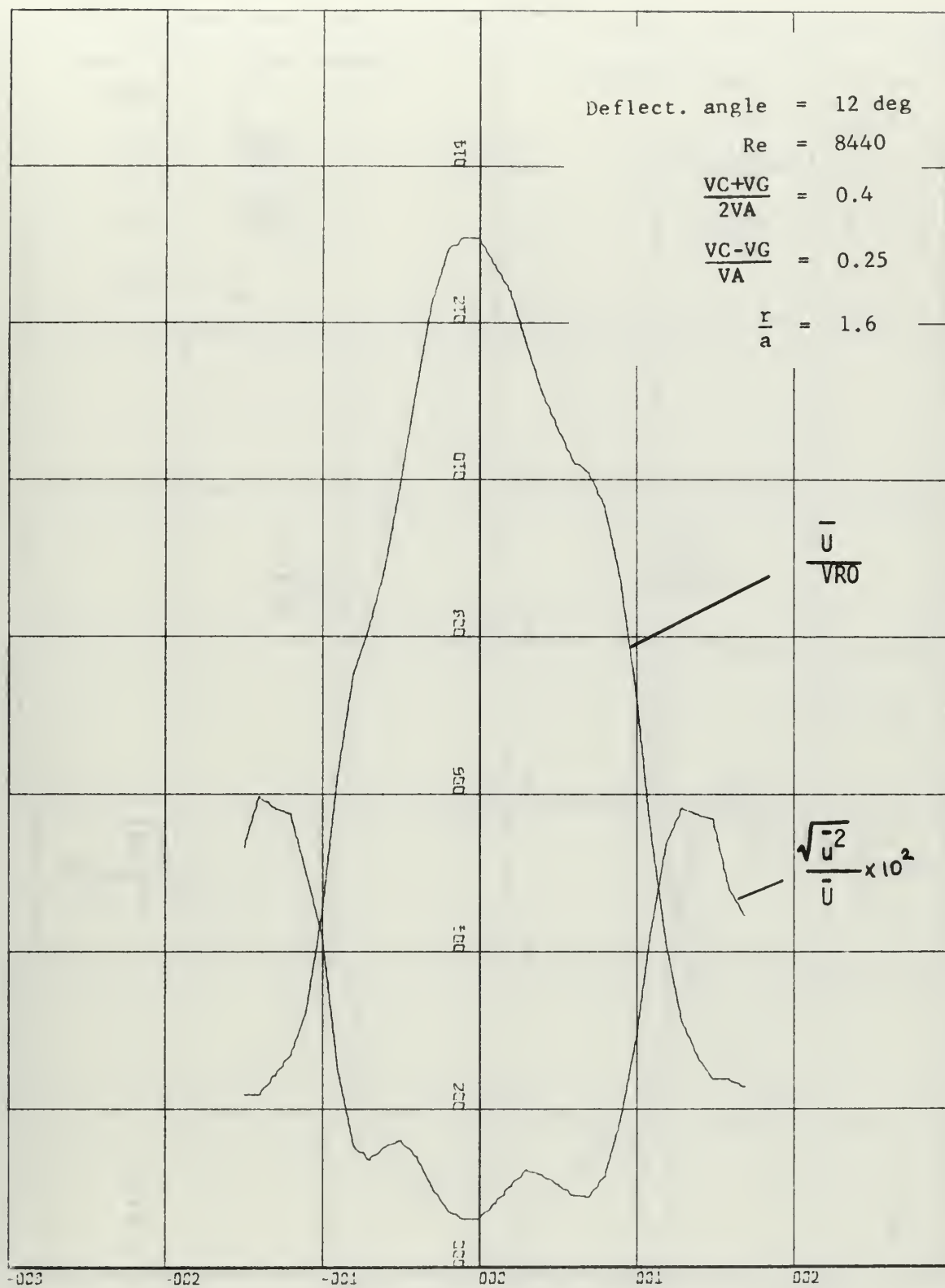


FIGURE 32 VELOCITY PROFILE AND TURBULENCE INTENSITY

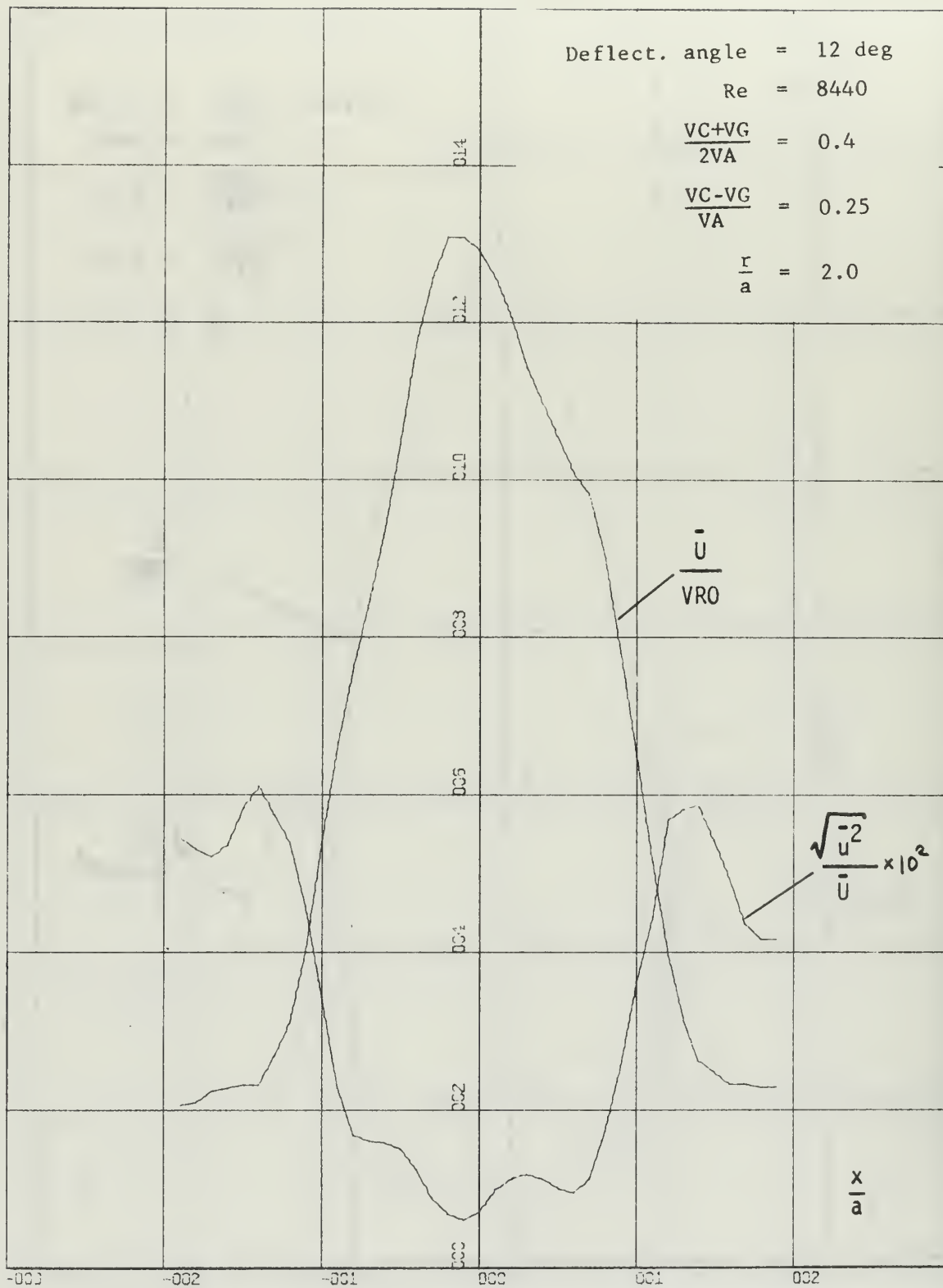


FIGURE 33 VELOCITY PROFILE AND TURBULENCE INTENSITY

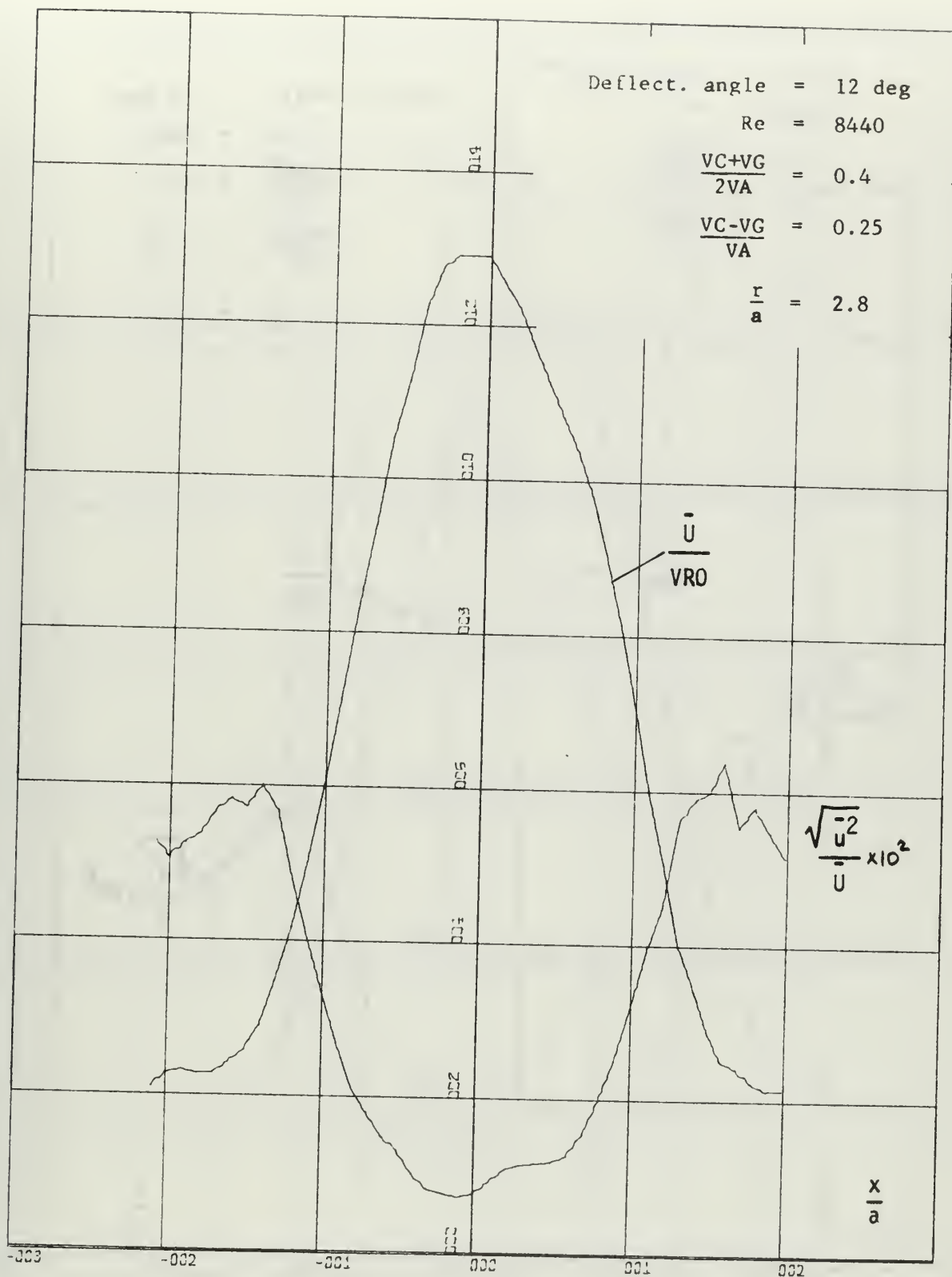


FIGURE 34 VELOCITY PROFILE AND TURBULENCE INTENSITY

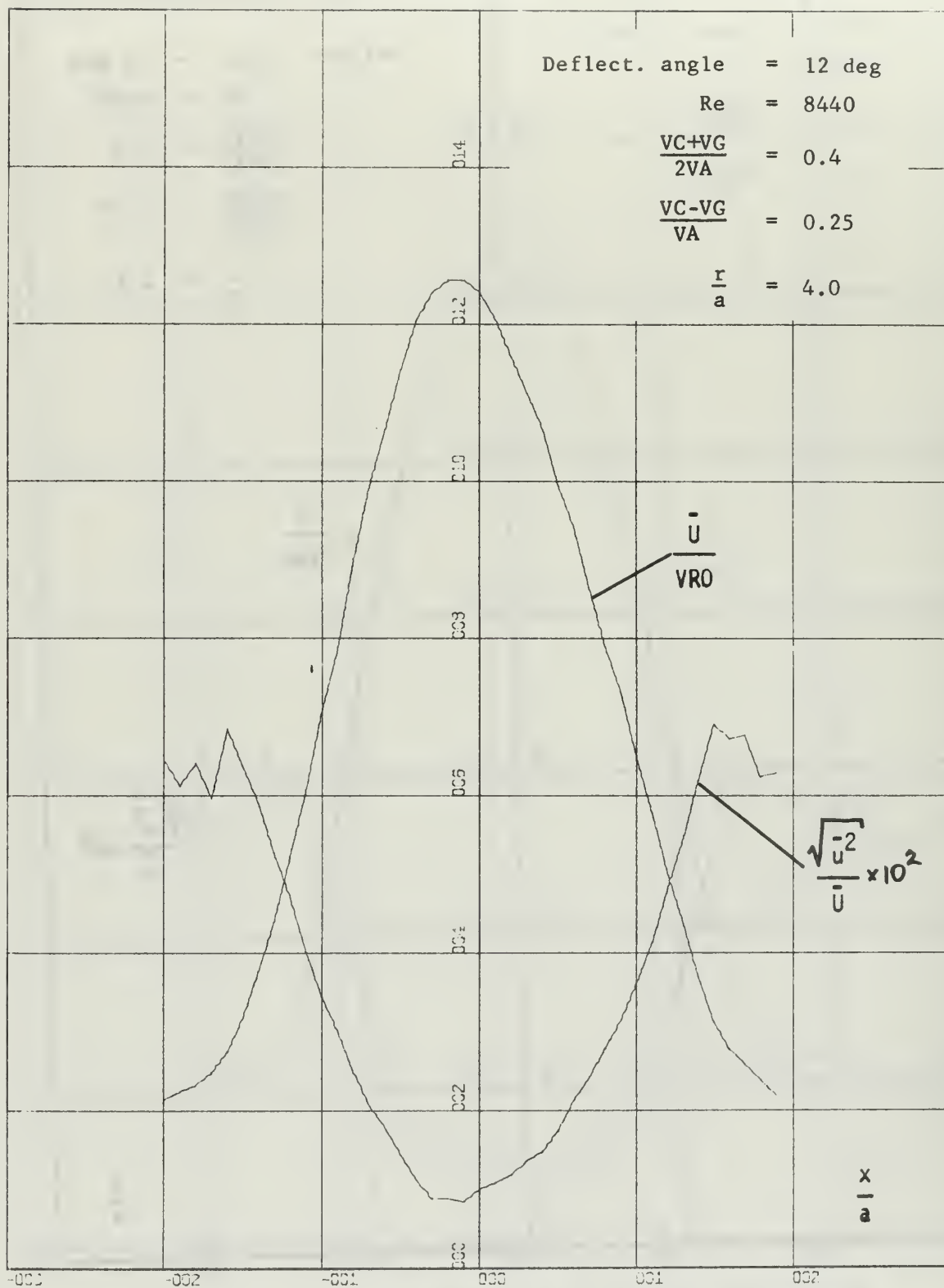


FIGURE 35 VELOCITY PROFILE AND TURBULENCE INTENSITY

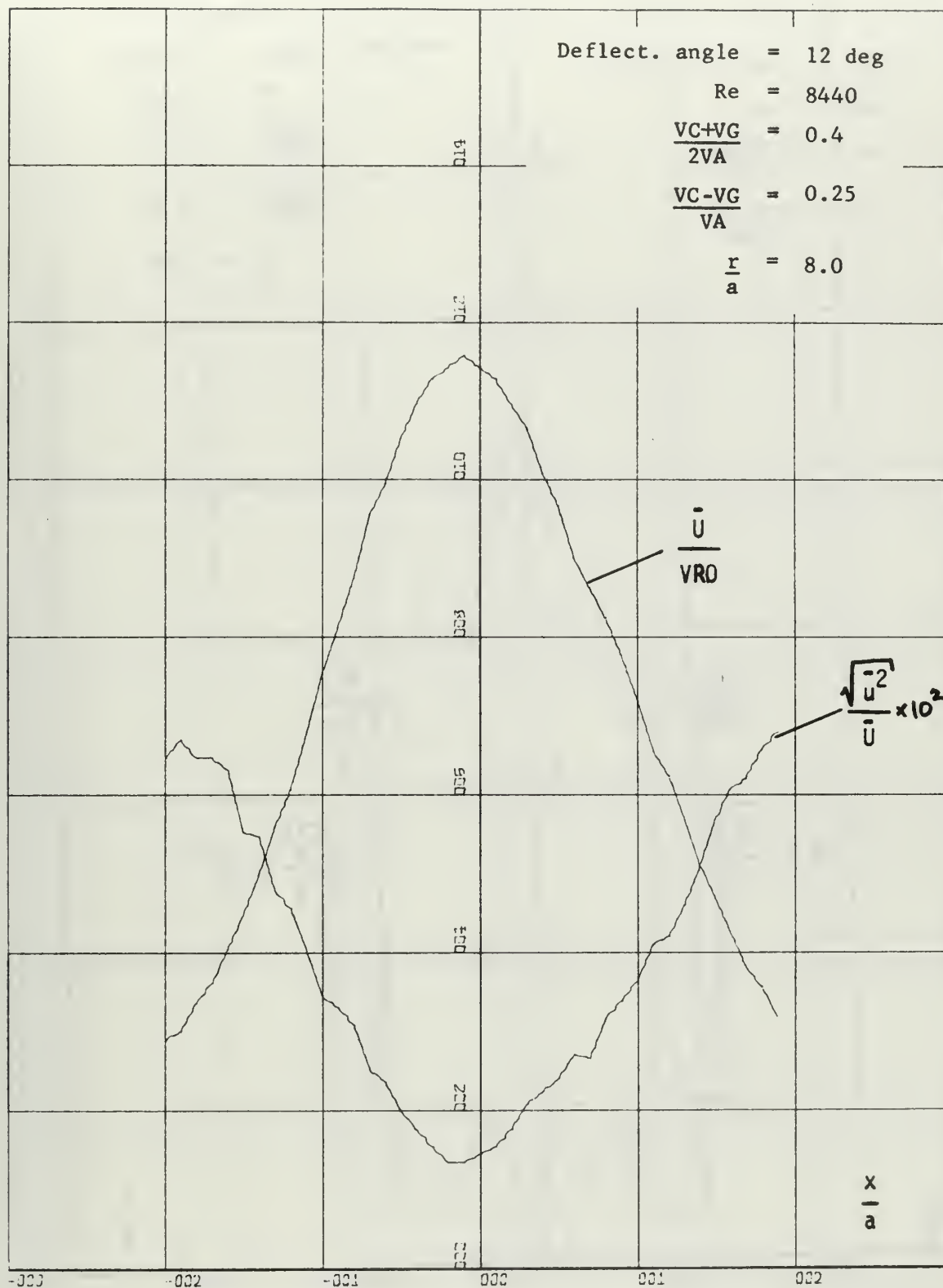


FIGURE 36 VELOCITY PROFILE AND TURBULENCE INTENSITY

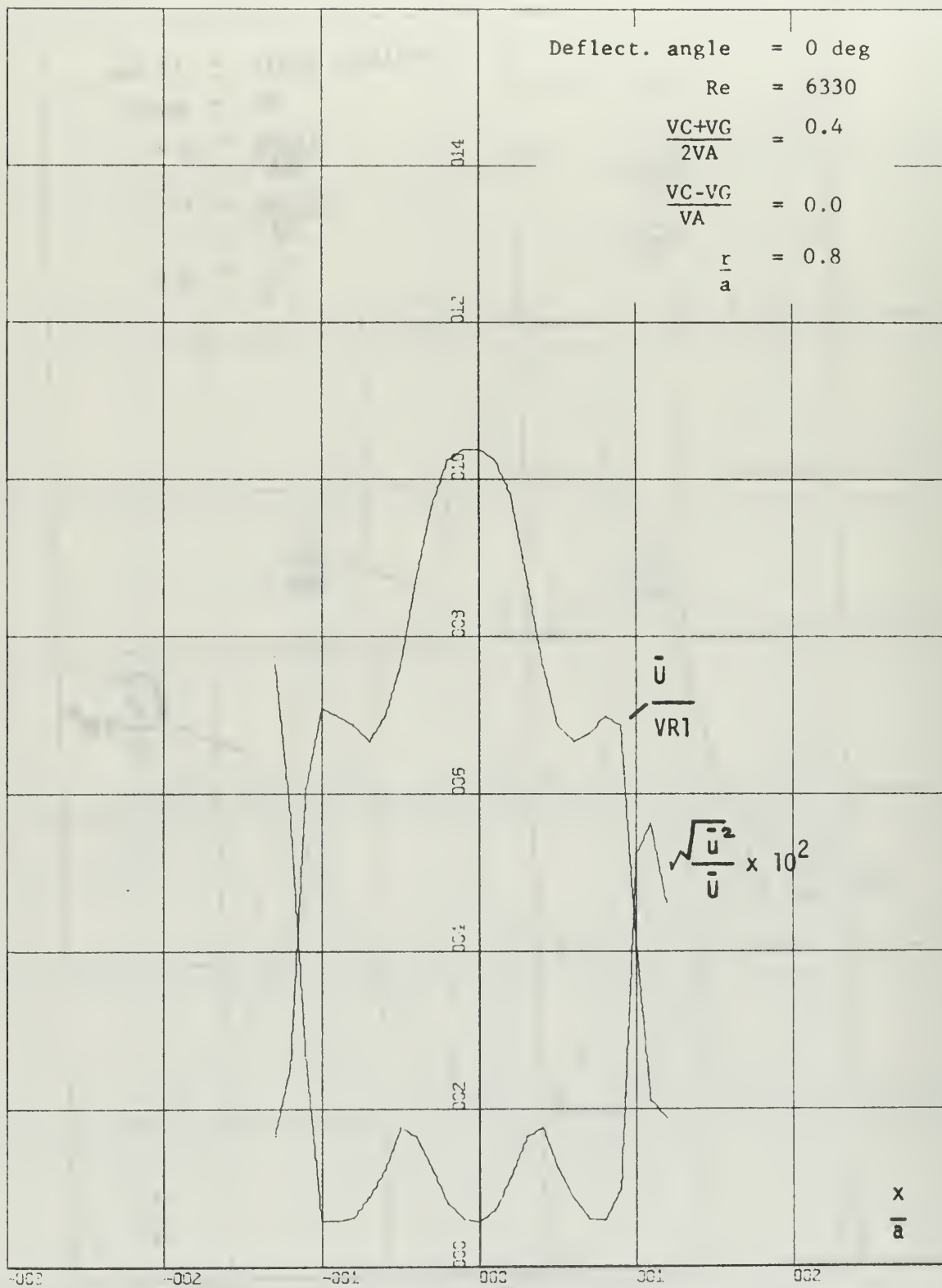


FIGURE 37 VELOCITY PROFILE AND TURBULENCE INTENSITY

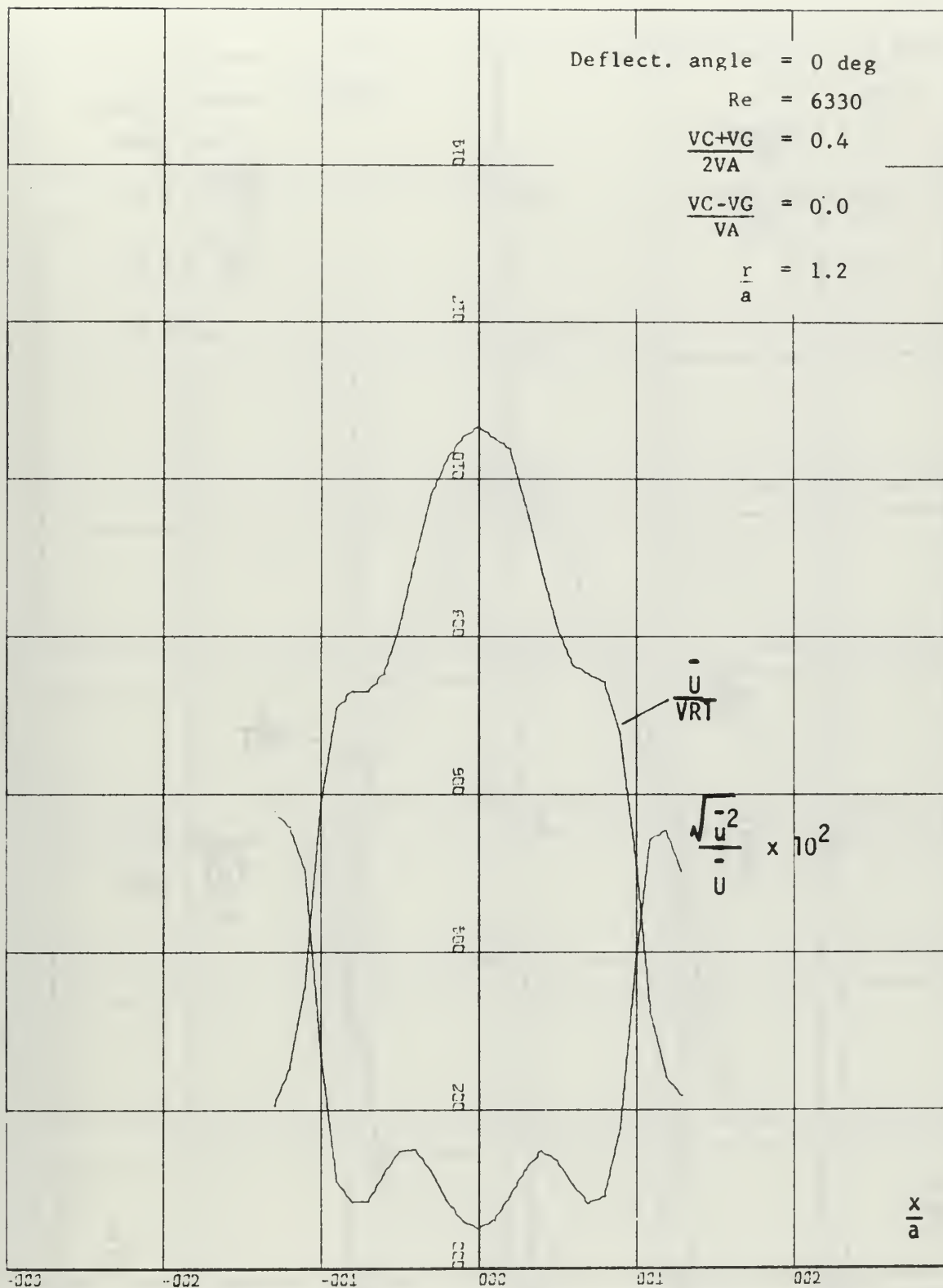


FIGURE 38 VELOCITY PROFILE AND TURBULENCE INTENSITY

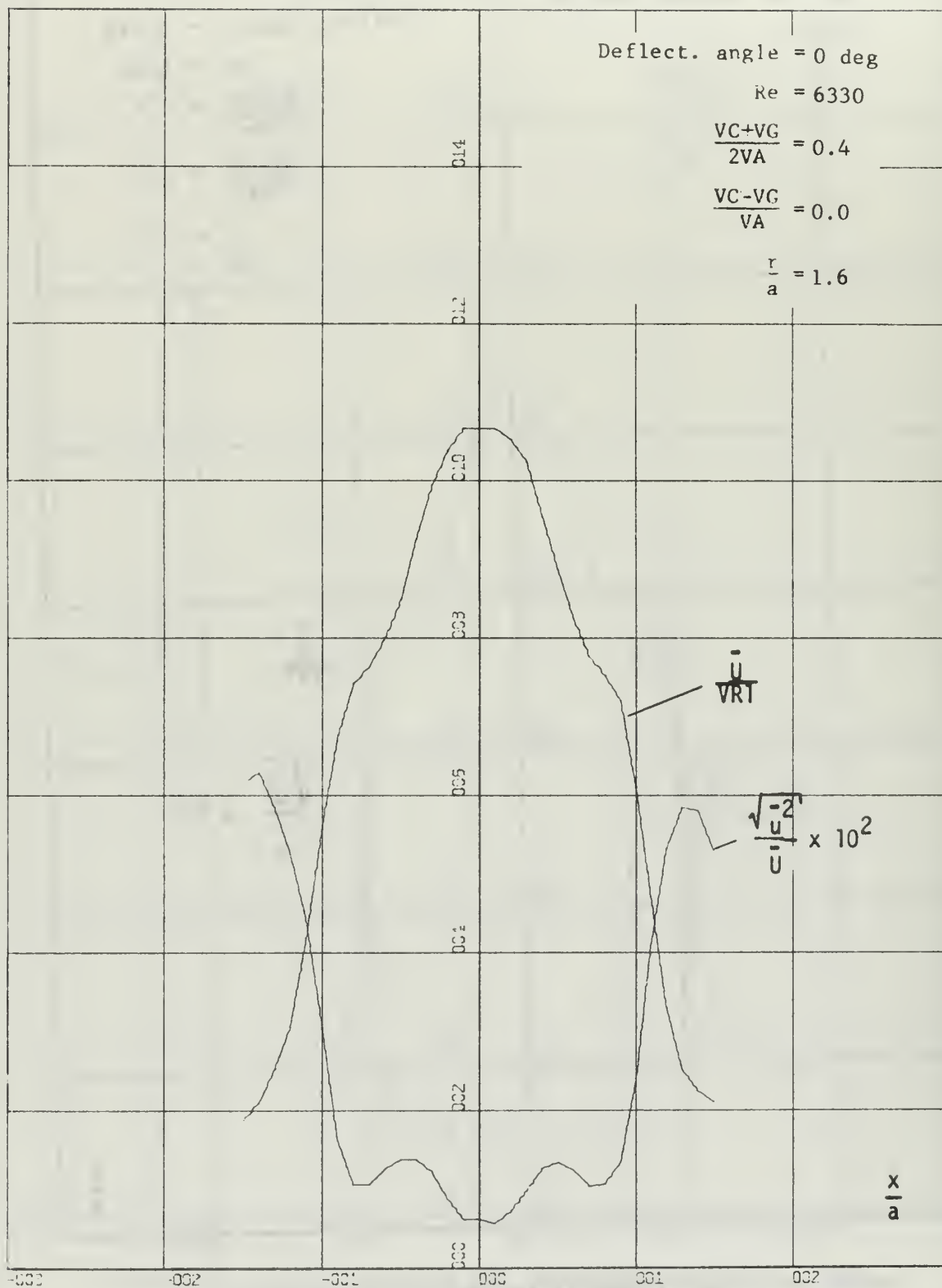


FIGURE 39 VELOCITY PROFILE AND TURBULENCE INTENSITY

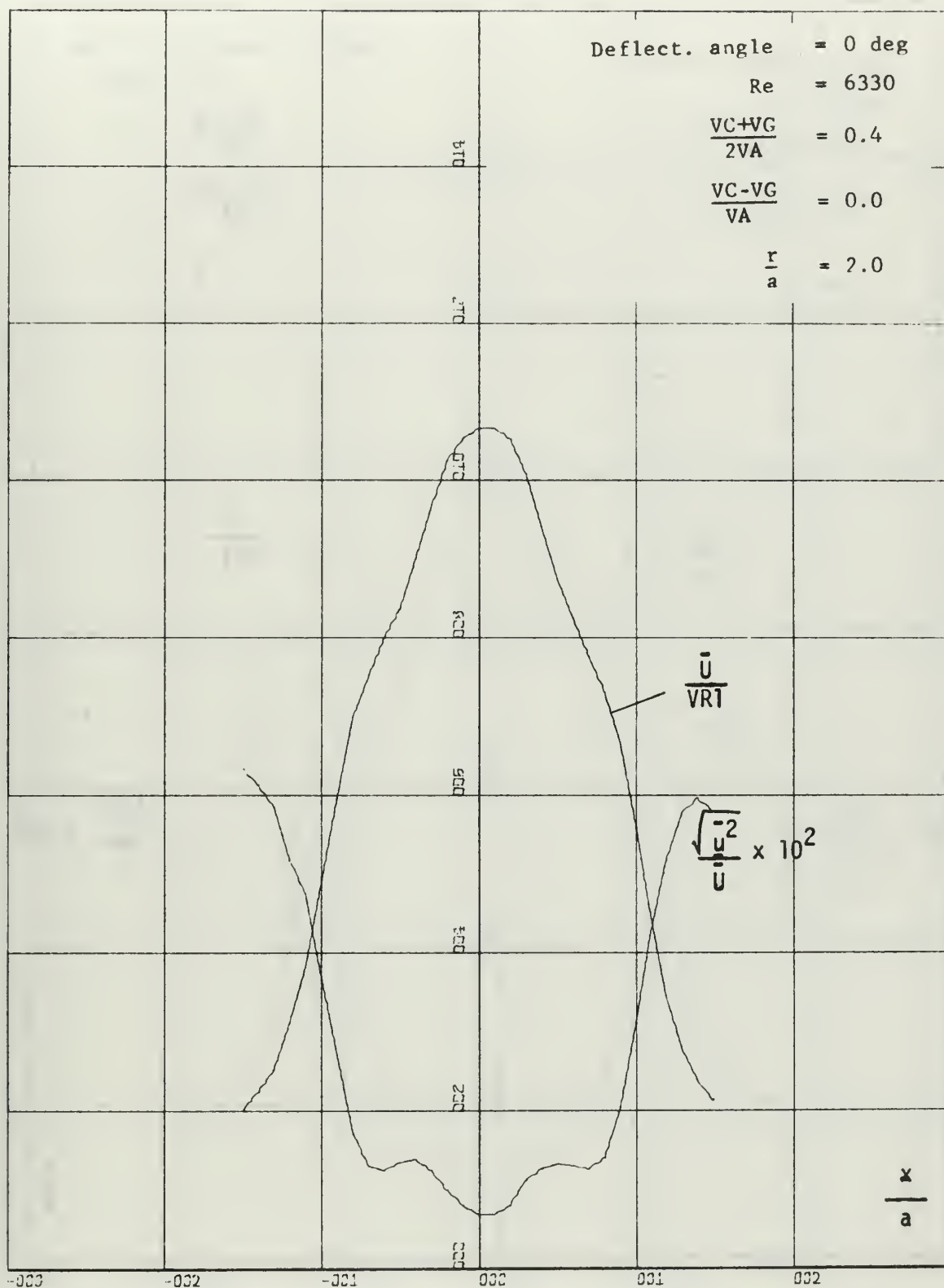


FIGURE 40 VELOCITY PROFILE AND TURBULENCE INTENSITY

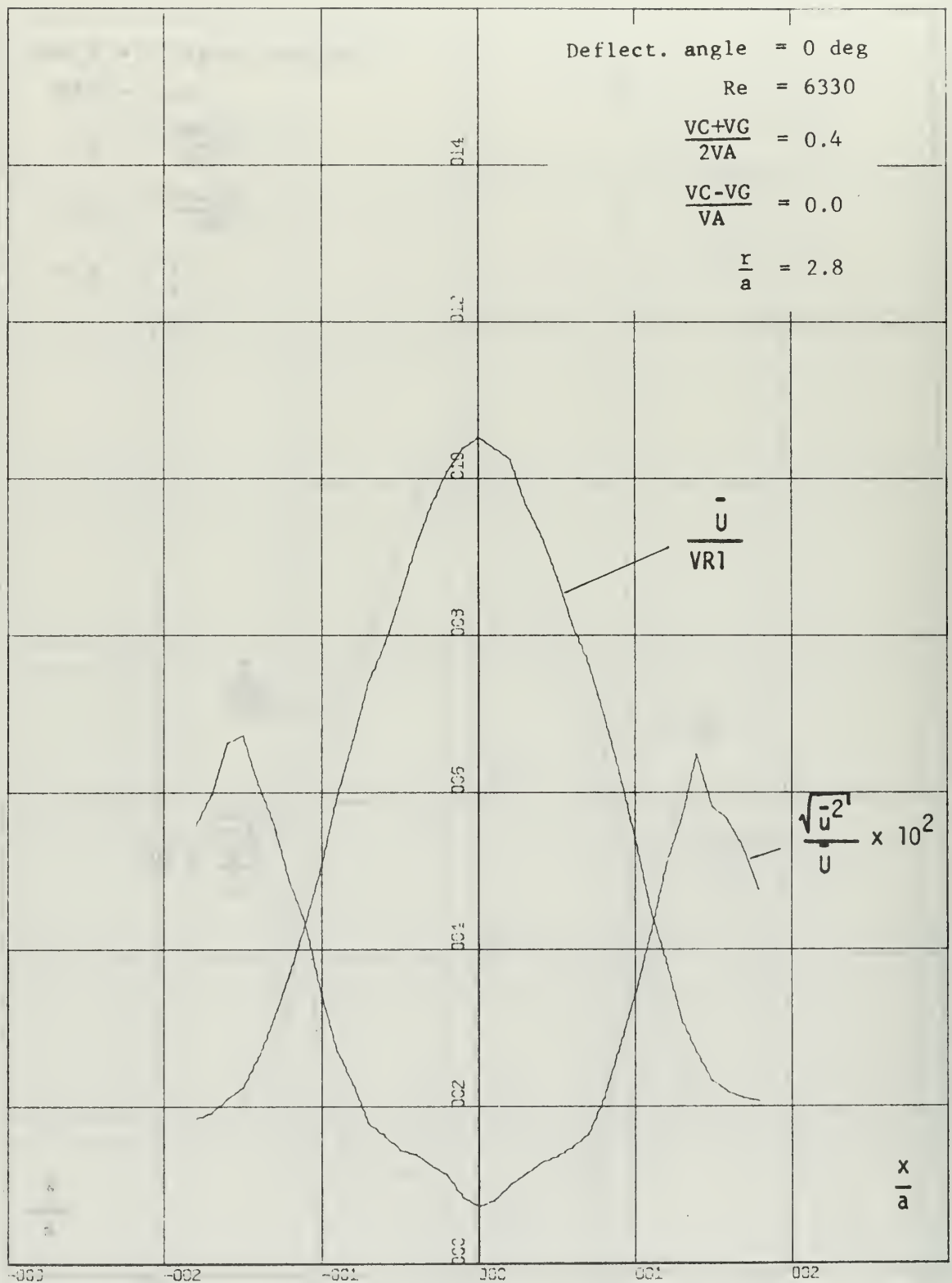


FIGURE 41 VELOCITY PROFILE AND TURBULENCE INTENSITY

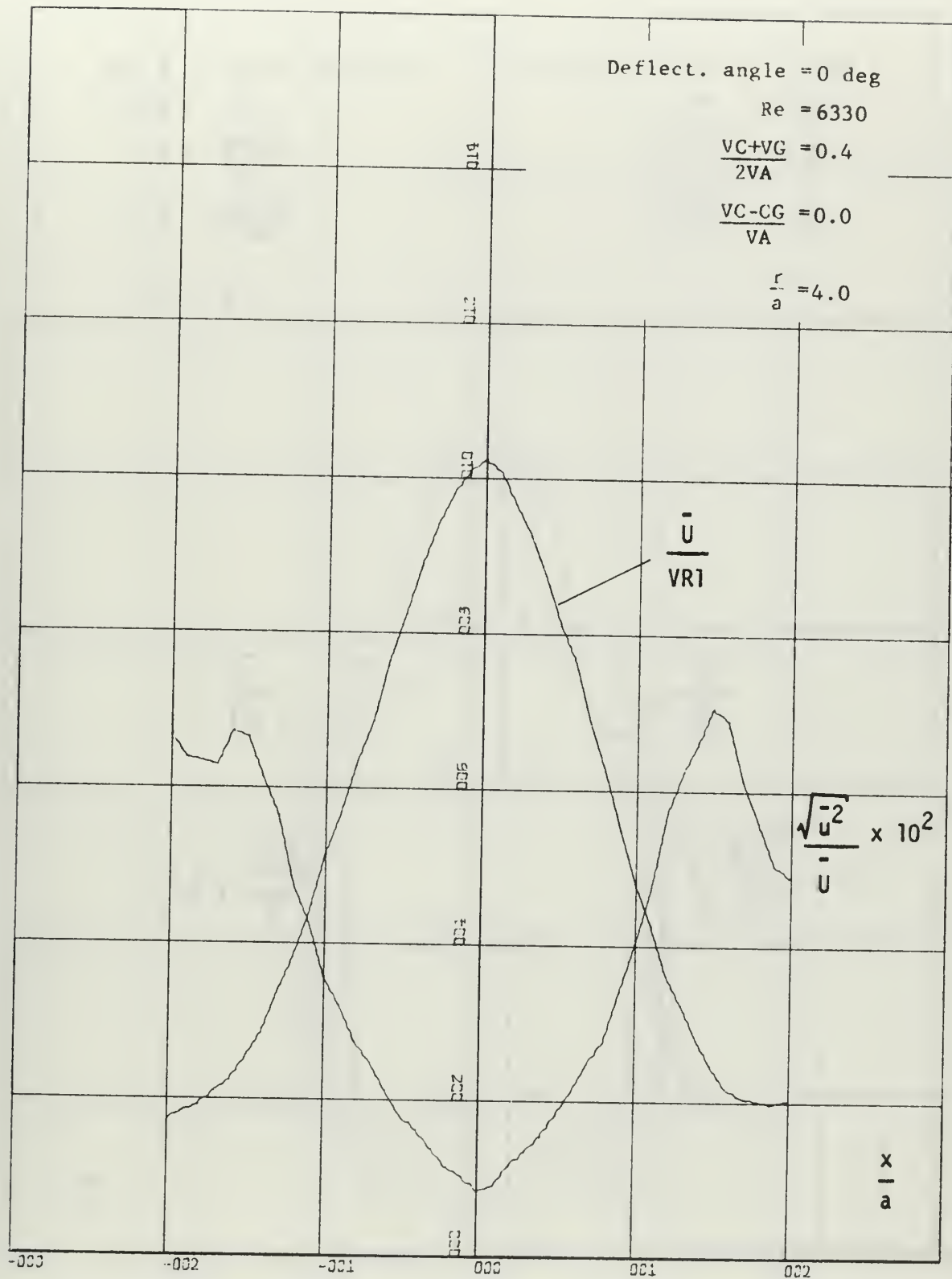


FIGURE 42 VELOCITY PROFILE AND TURBULENCE INTENSITY

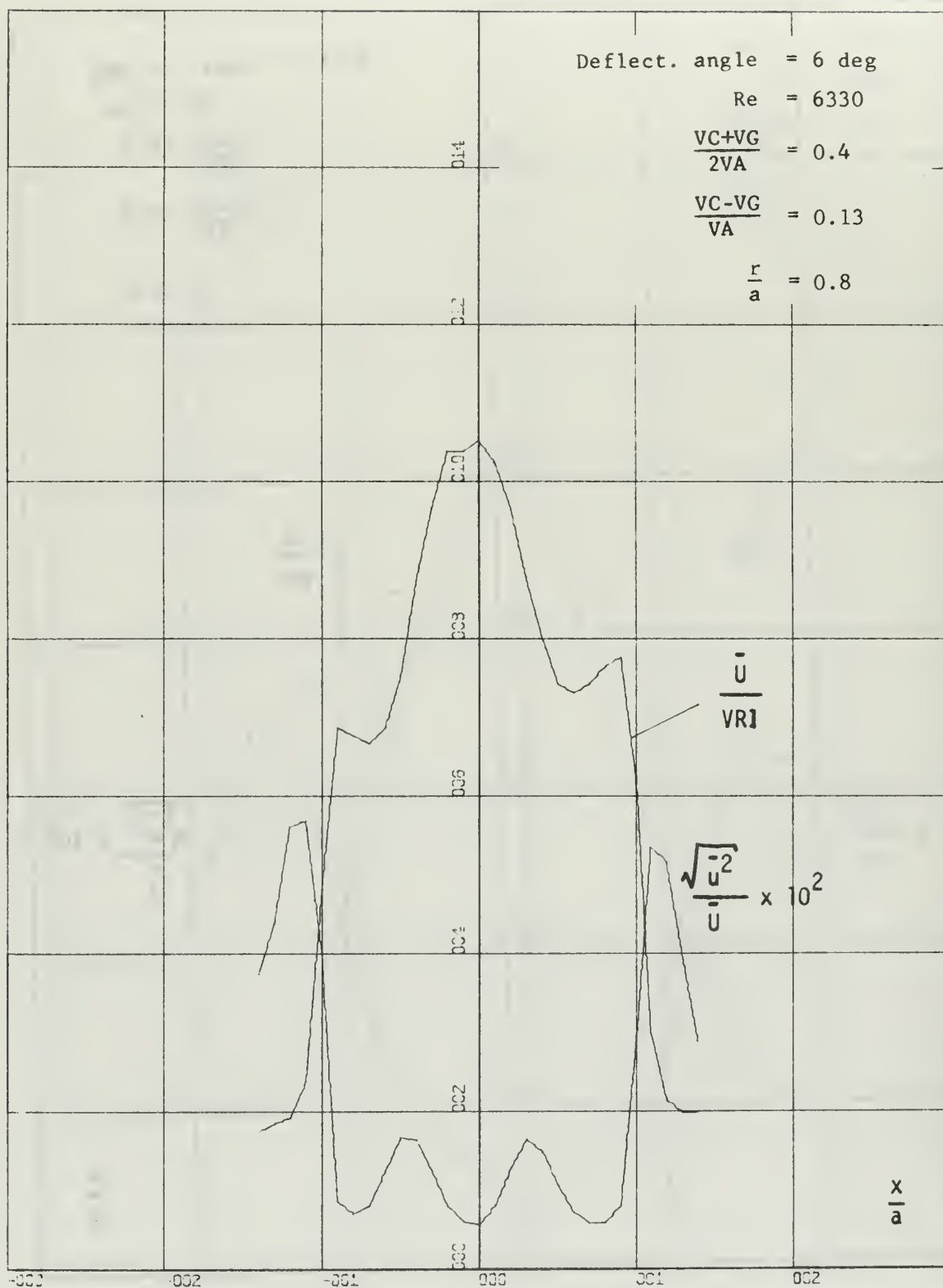


FIGURE 43 VELOCITY PROFILE AND TURBULENCE INTENSITY

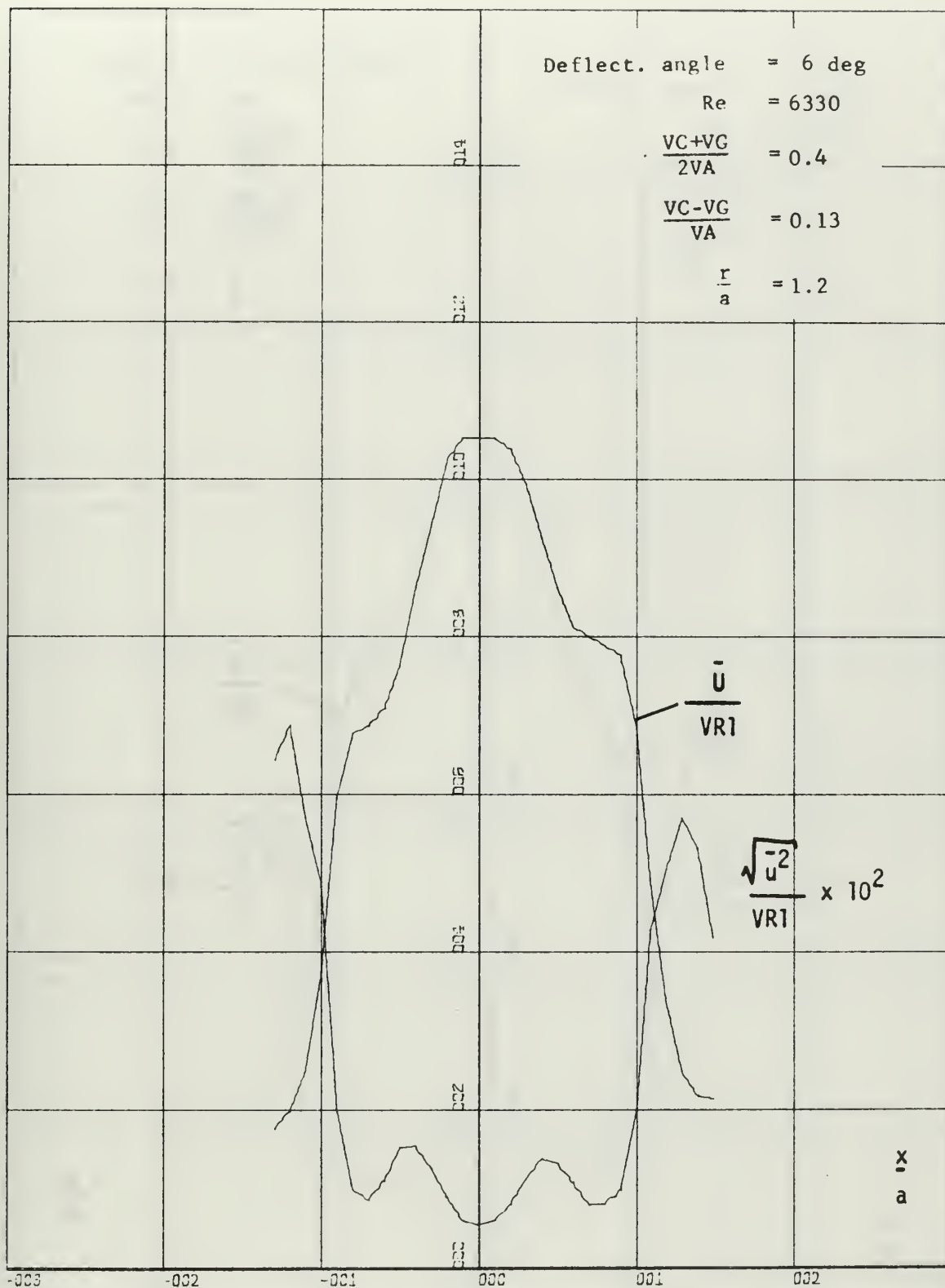


FIGURE 44 VELOCITY PROFILE AND TURBULENCE INTENSITY

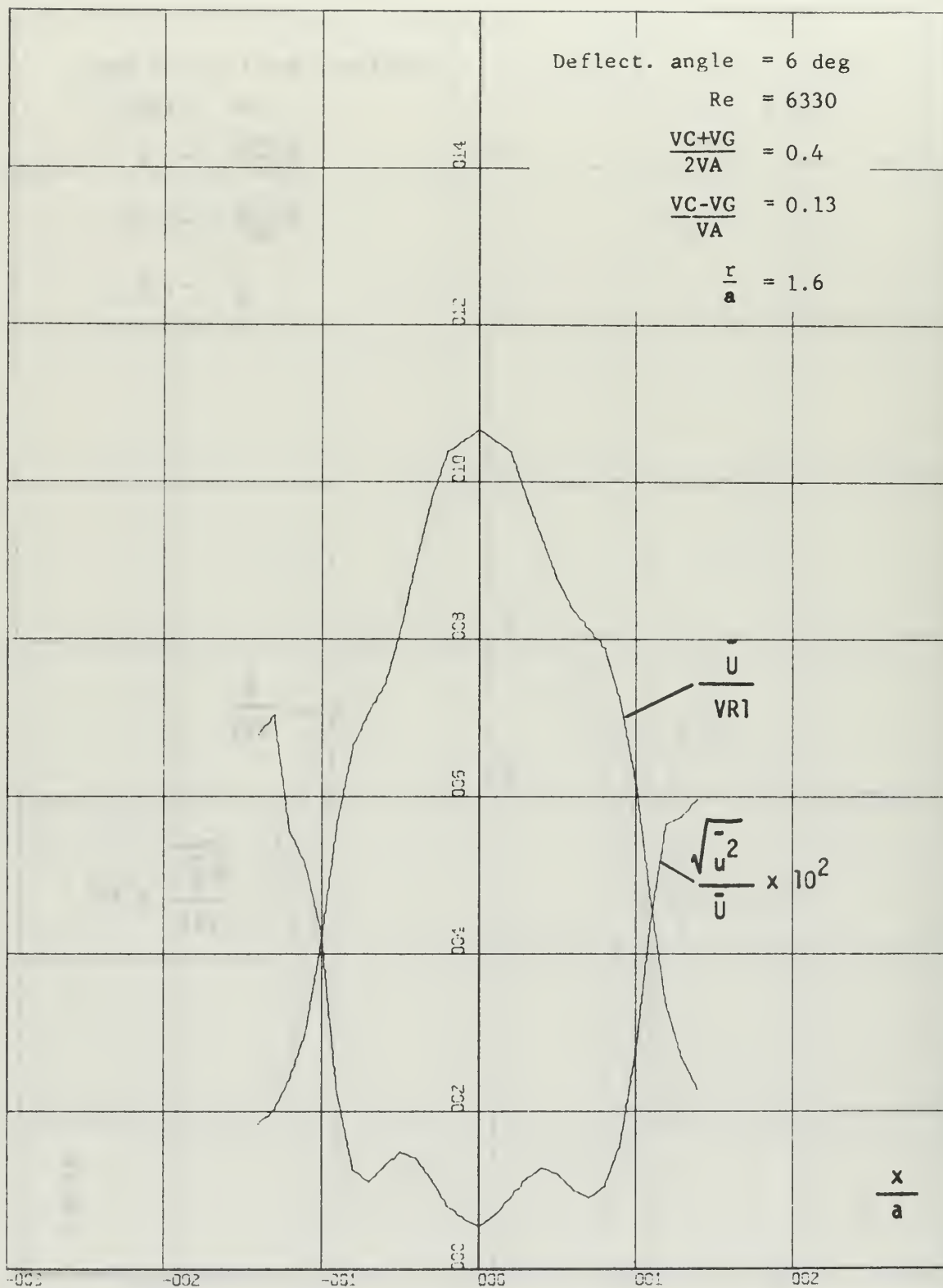


FIGURE 45 VELOCITY PROFILE AND TURBULENCE INTENSITY

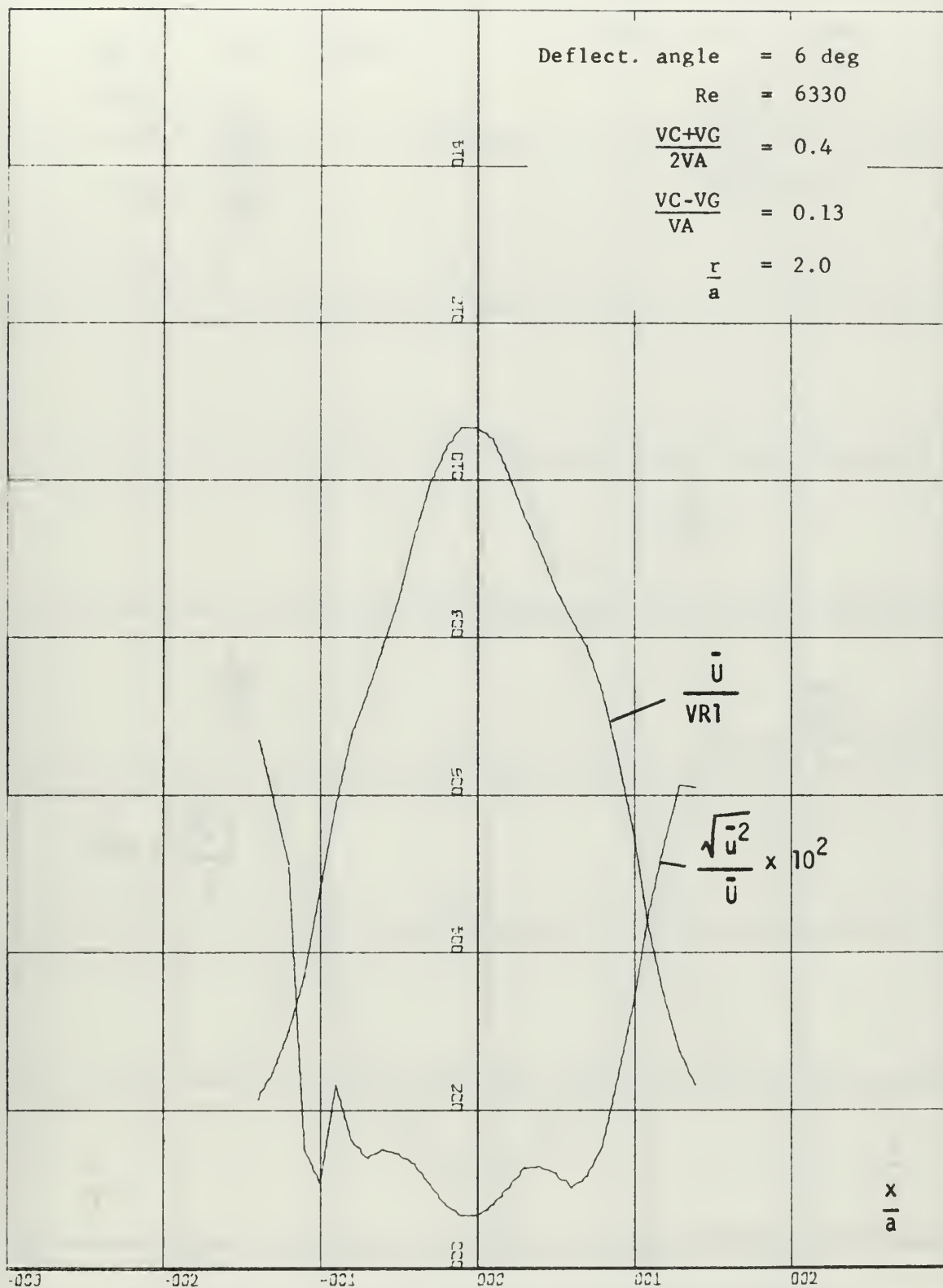


FIGURE 46 VELOCITY PROFILE AND TURBULENCE INTENSITY

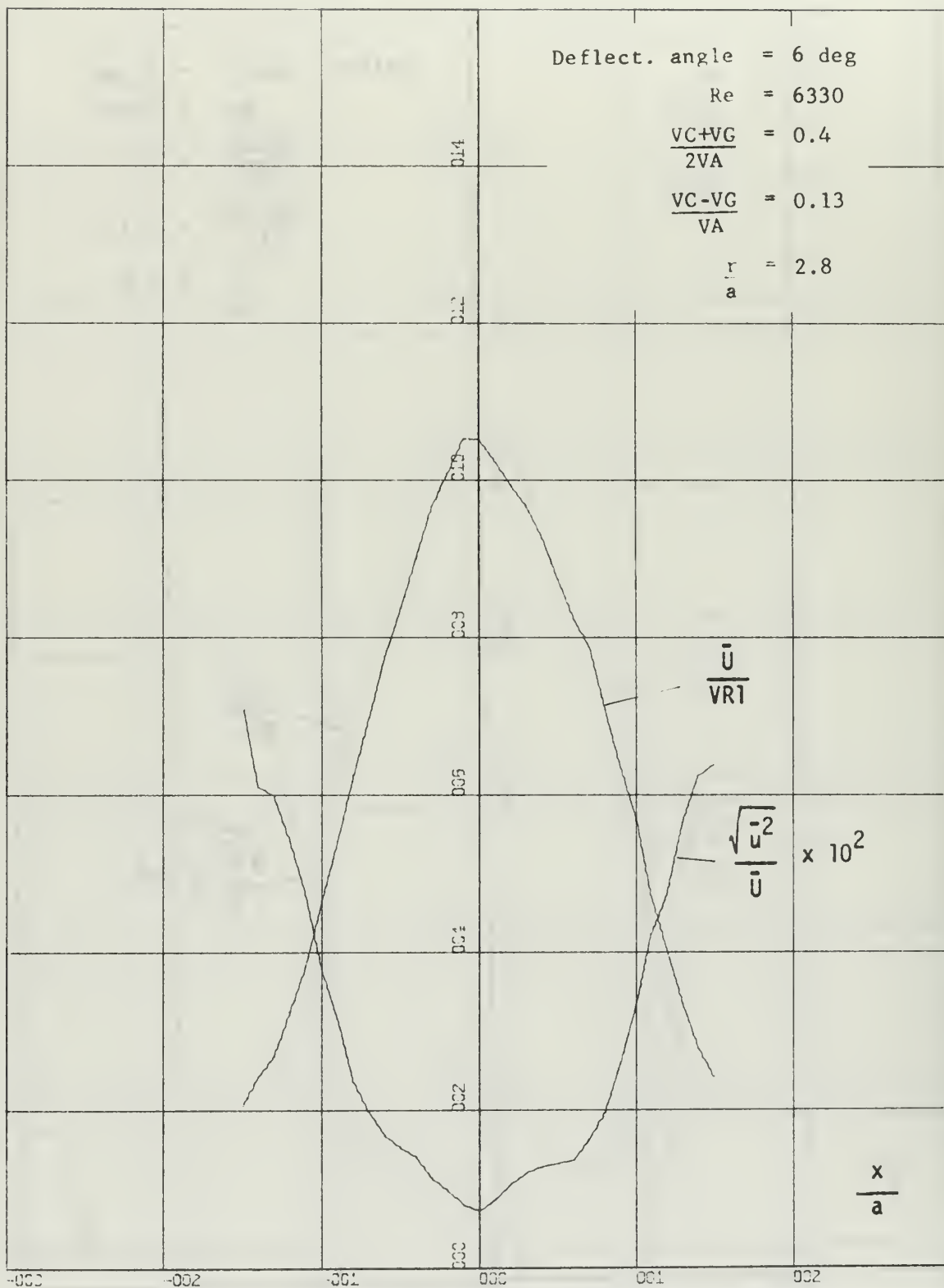


FIGURE 47 VELOCITY PROFILE AND TURBULENCE INTENSITY

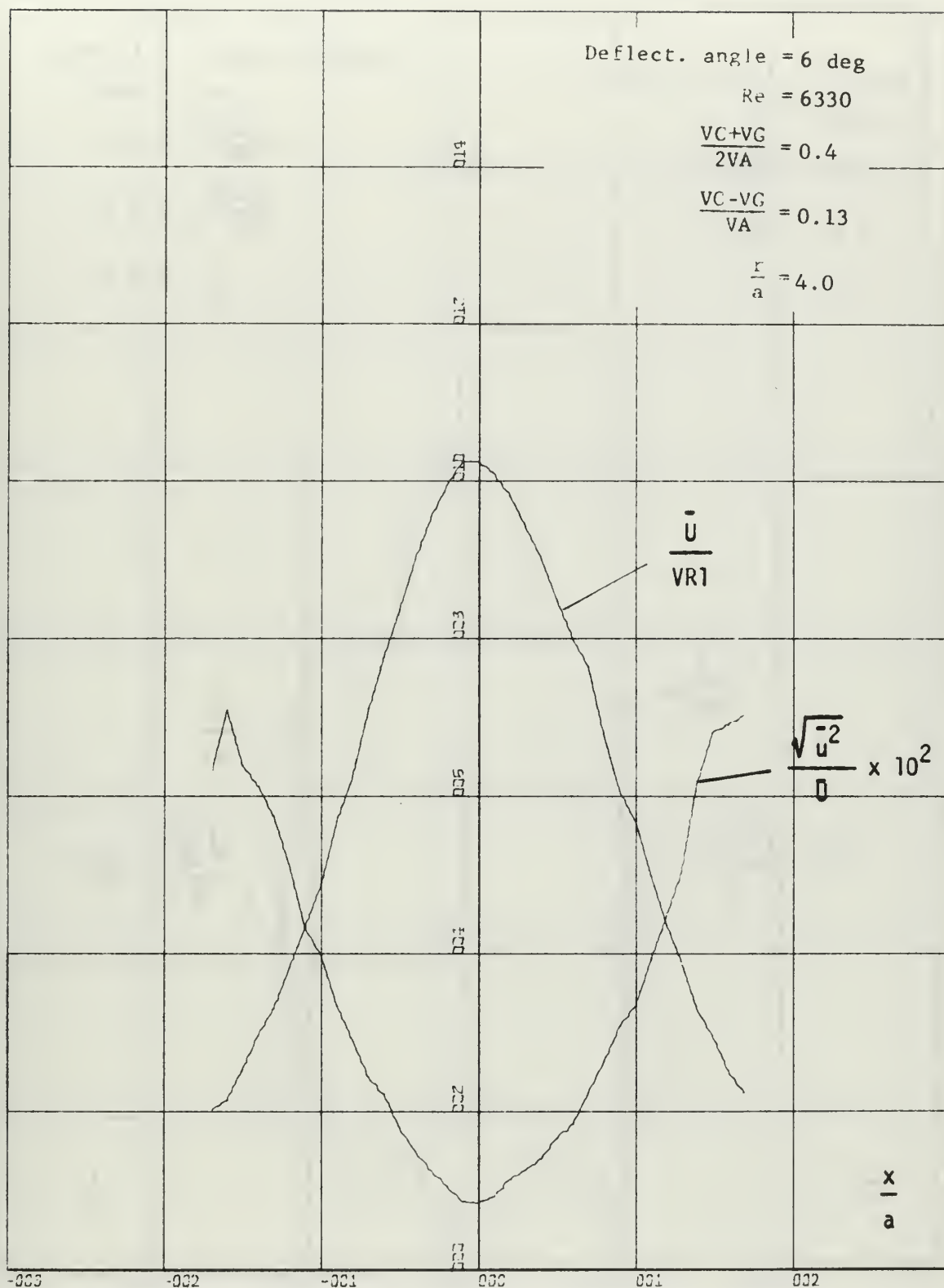


FIGURE 48 VELOCITY PROFILE AND TURBULENCE INTENSITY

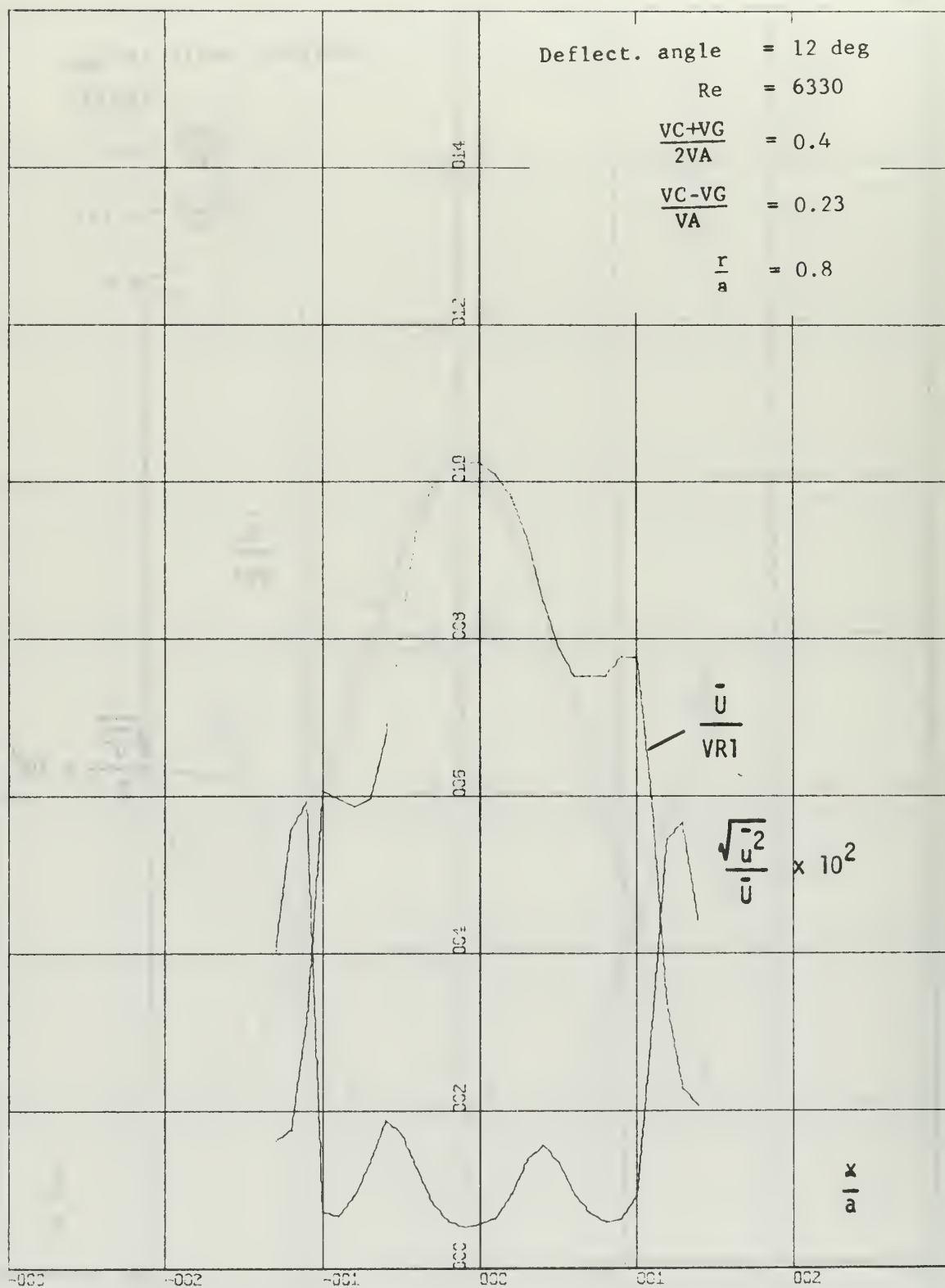


FIGURE 49 VELOCITY PROFILE AND TURBULENCE INTENSITY

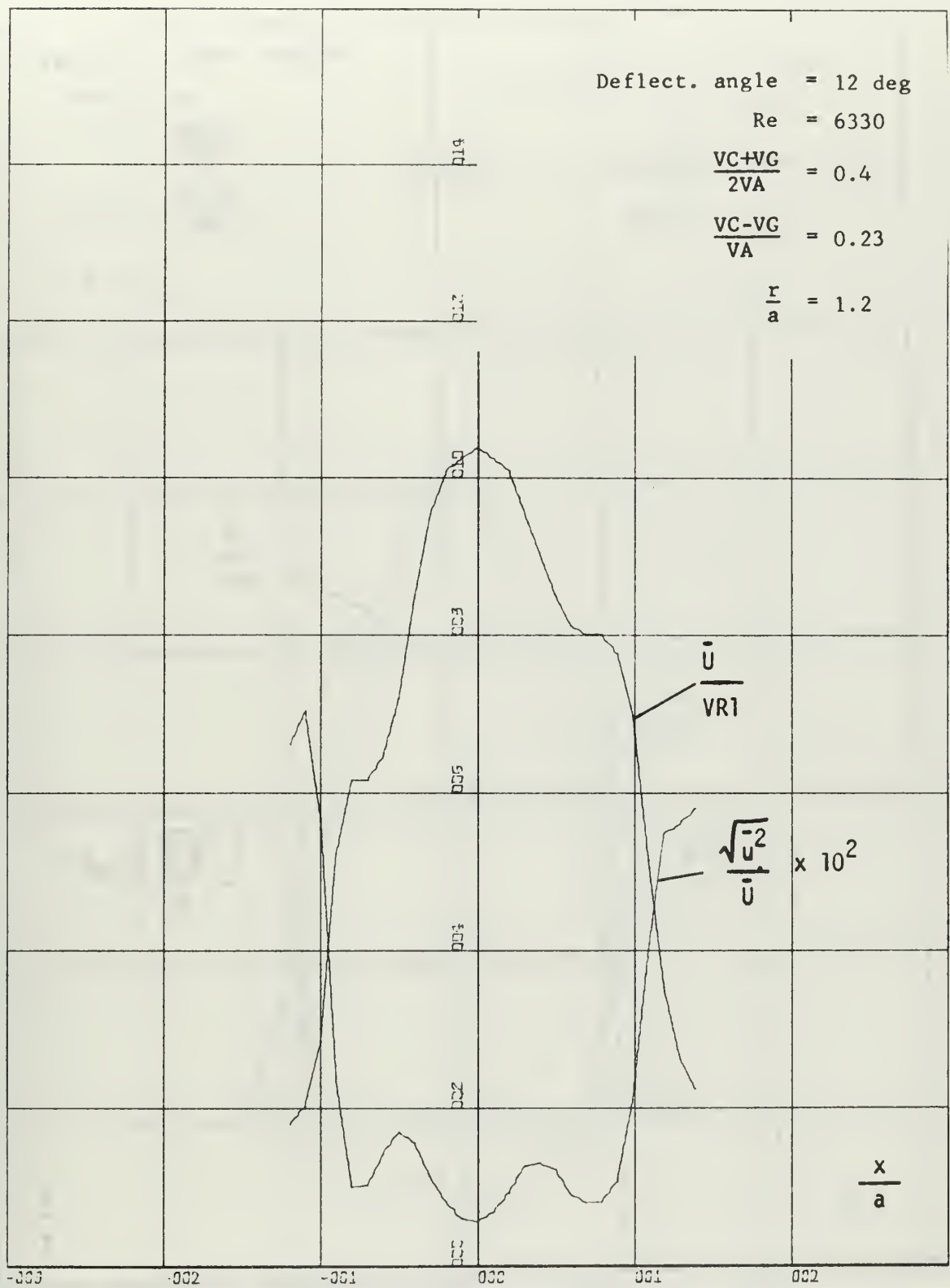


FIGURE 50 VELOCITY PROFILE AND TURBULENCE INTENSITY

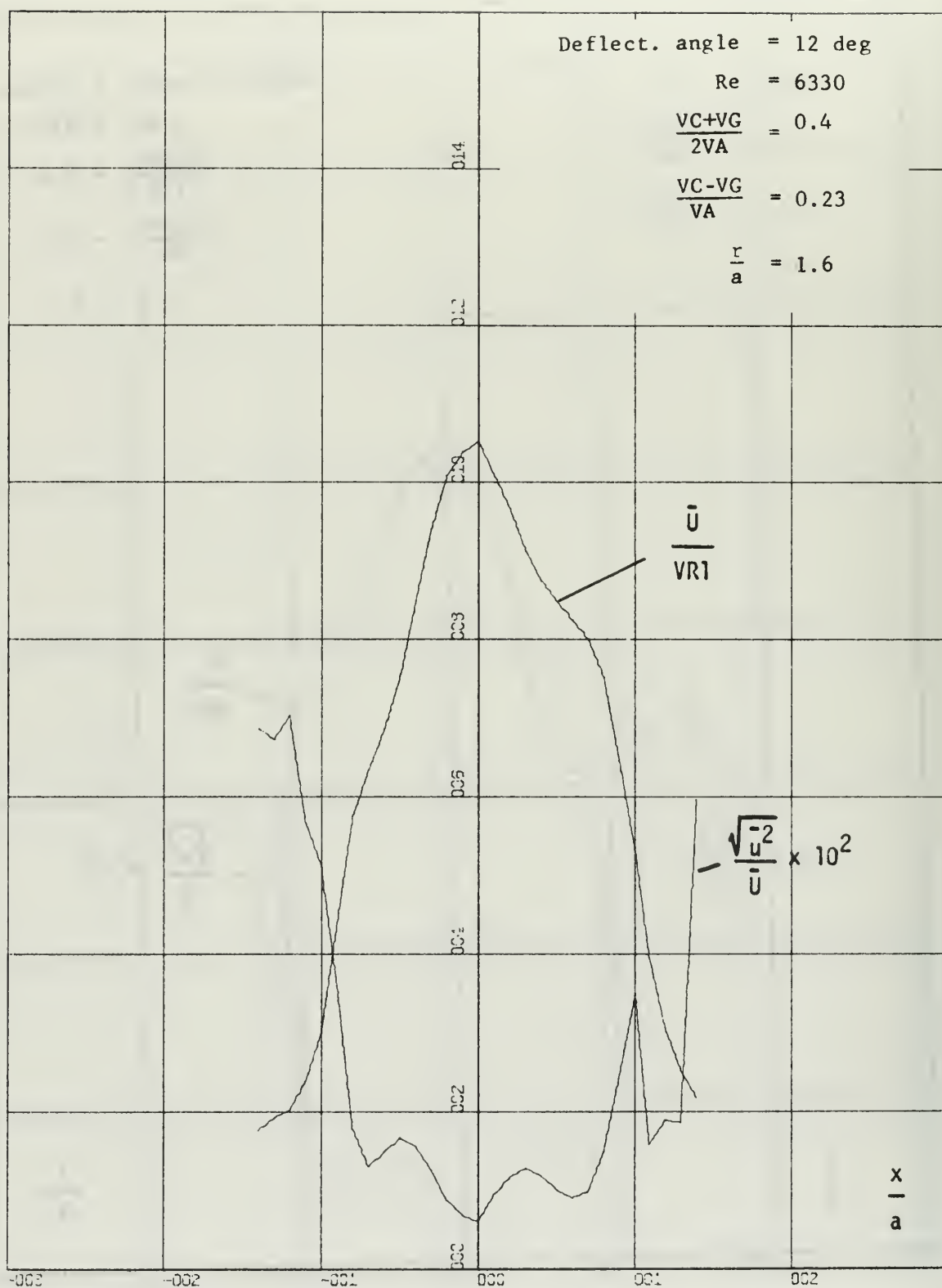


FIGURE 51 VELOCITY PROFILE AND TURBULENCE INTENSITY

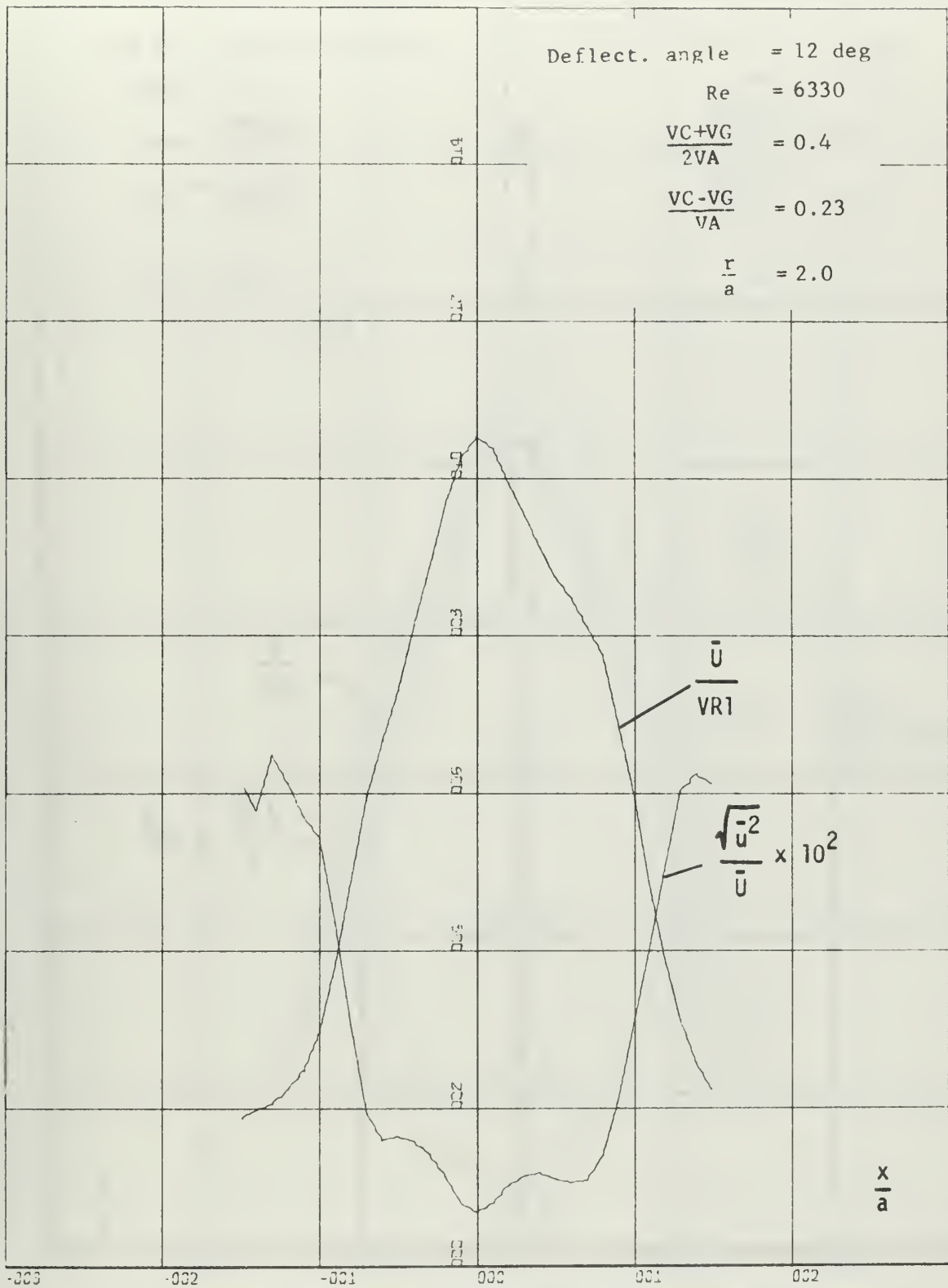


FIGURE 52 VELOCITY PROFILE AND TURBULENCE INTENSITY

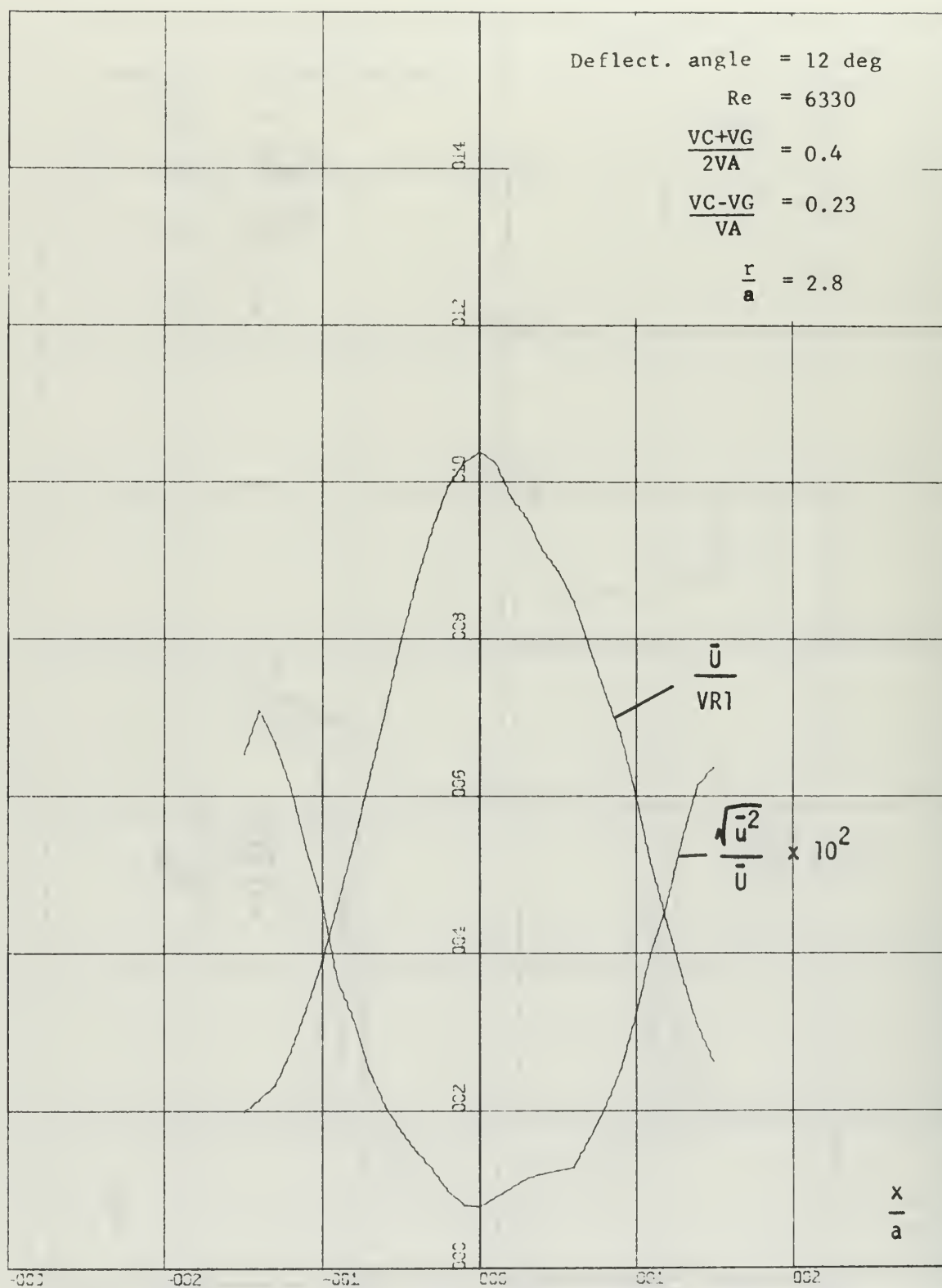


FIGURE 53 VELOCITY PROFILE AND TURBULENCE INTENSITY

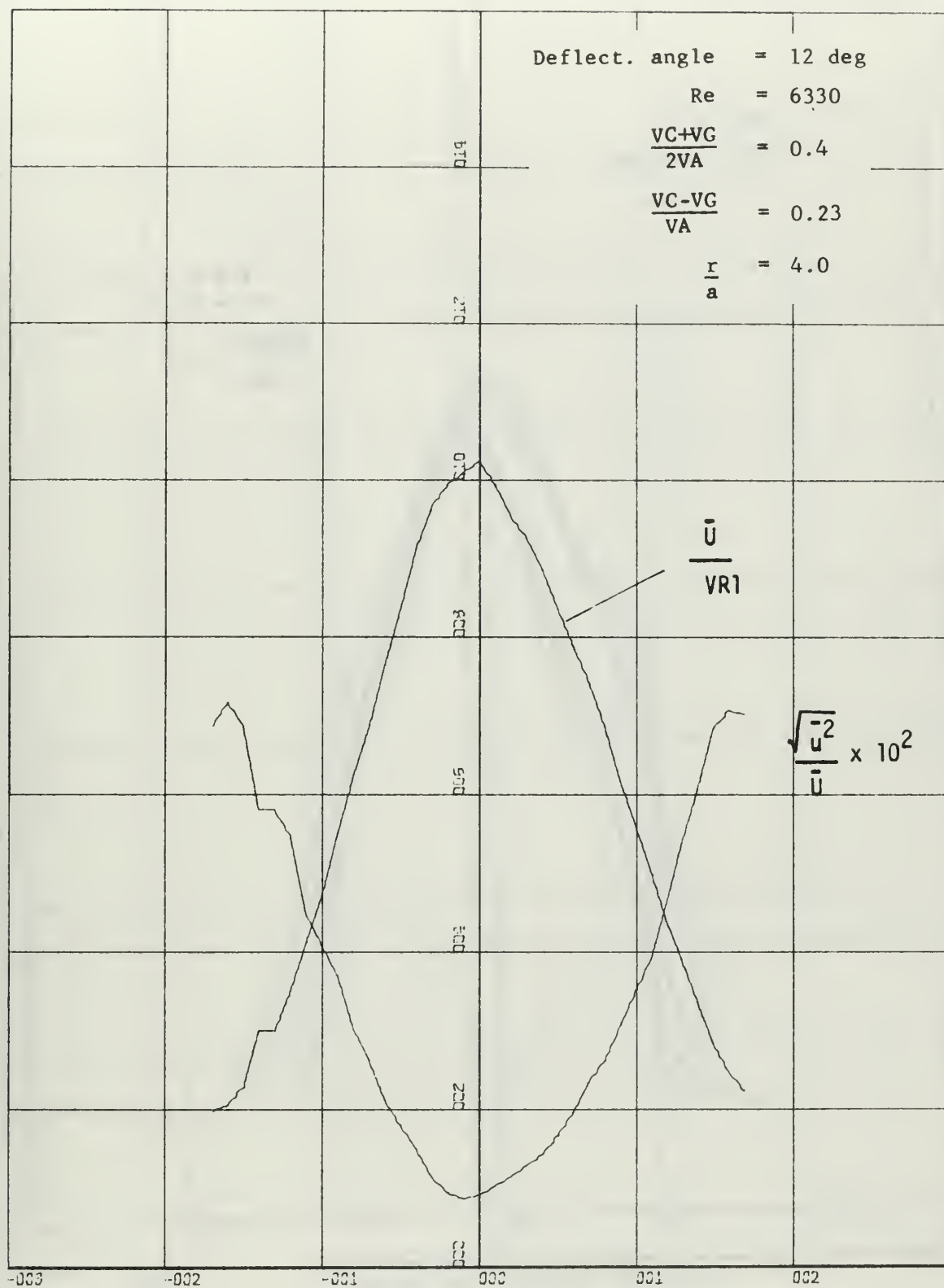


FIGURE 54 VELOCITY PROFILE AND TURBULENCE INTENSITY

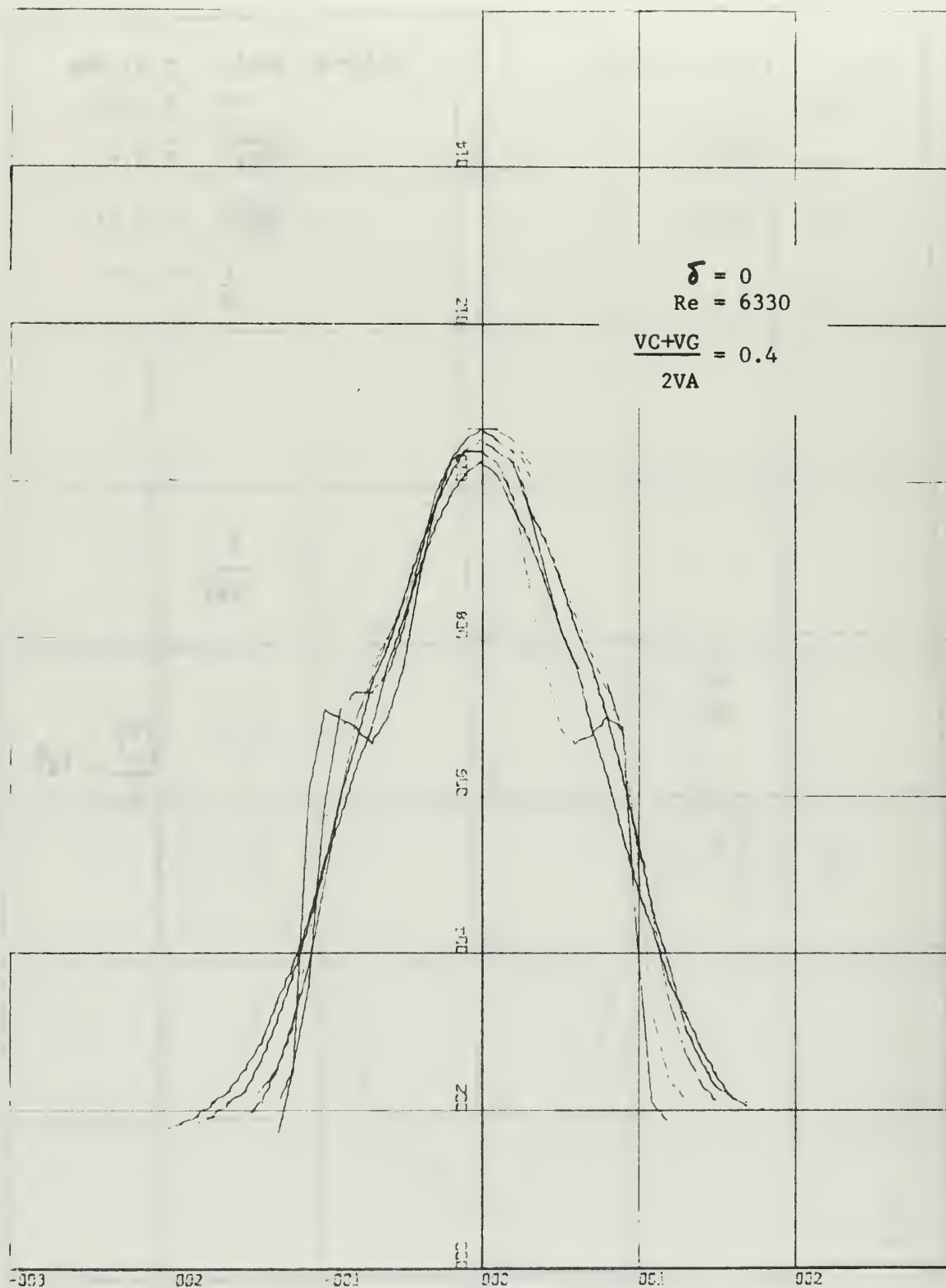


FIGURE 55 VELOCITY PROFILES

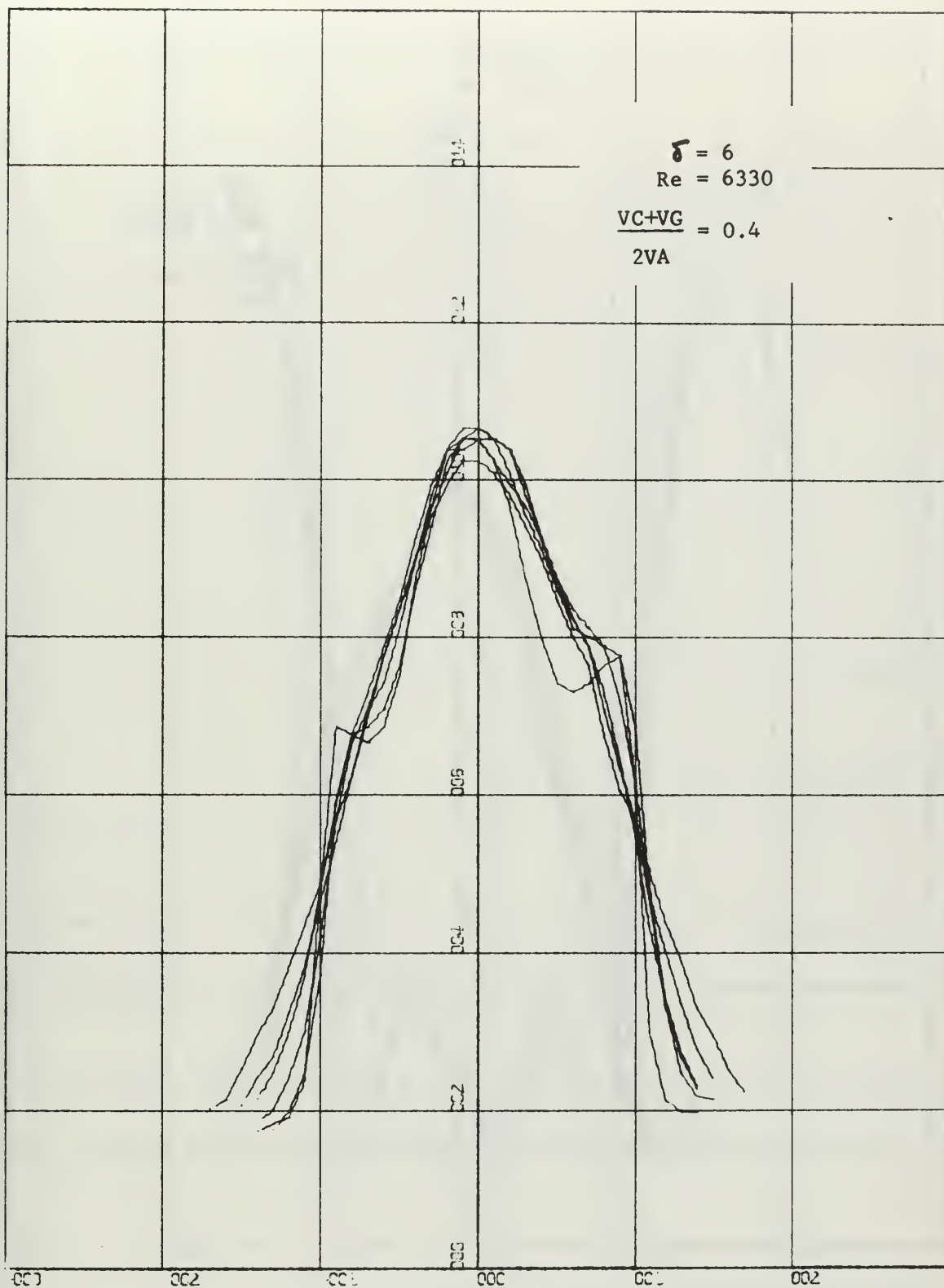


FIGURE 56 VELOCITY PROFILES

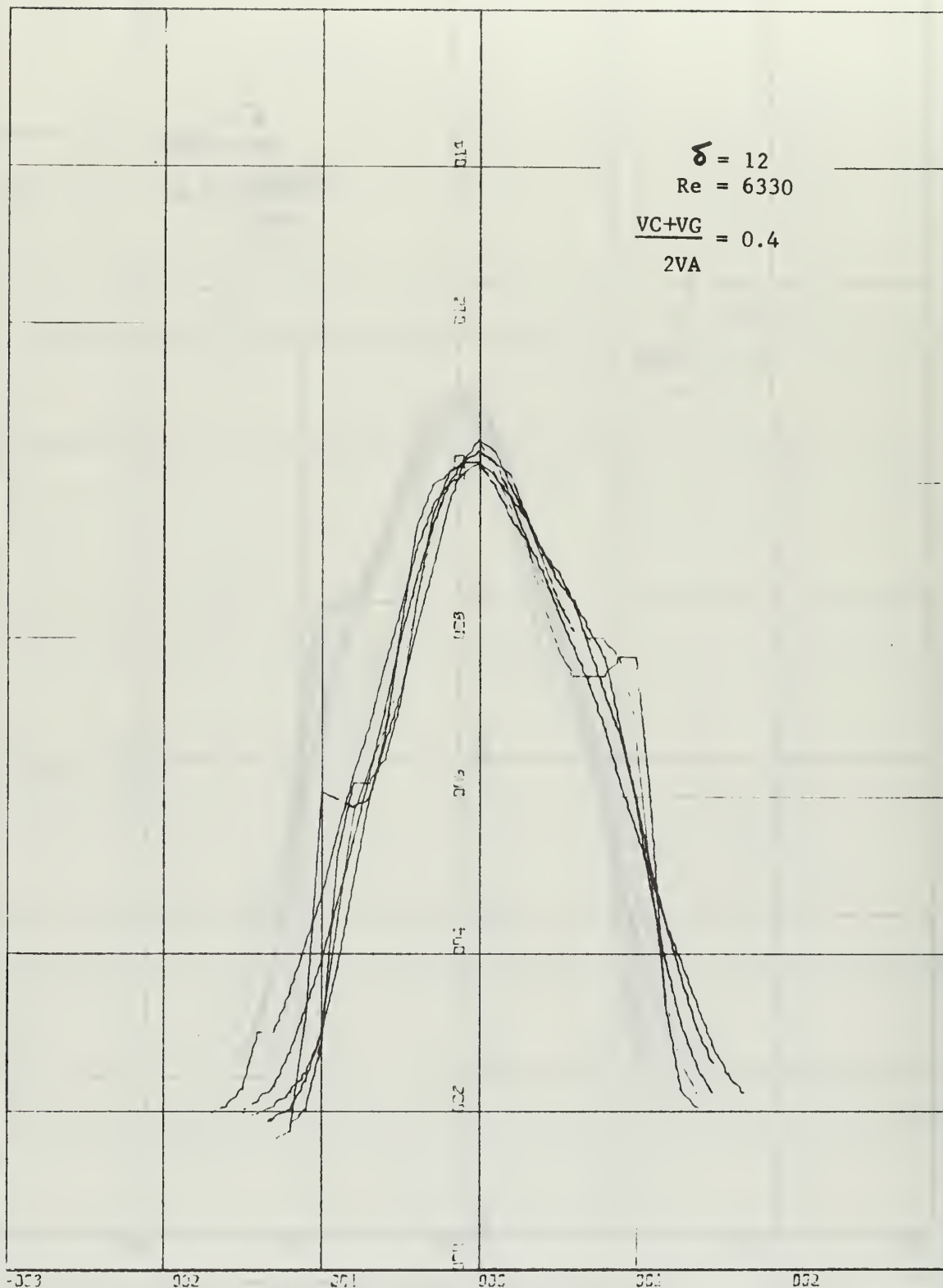


FIGURE 57 VELOCITY PROFILES

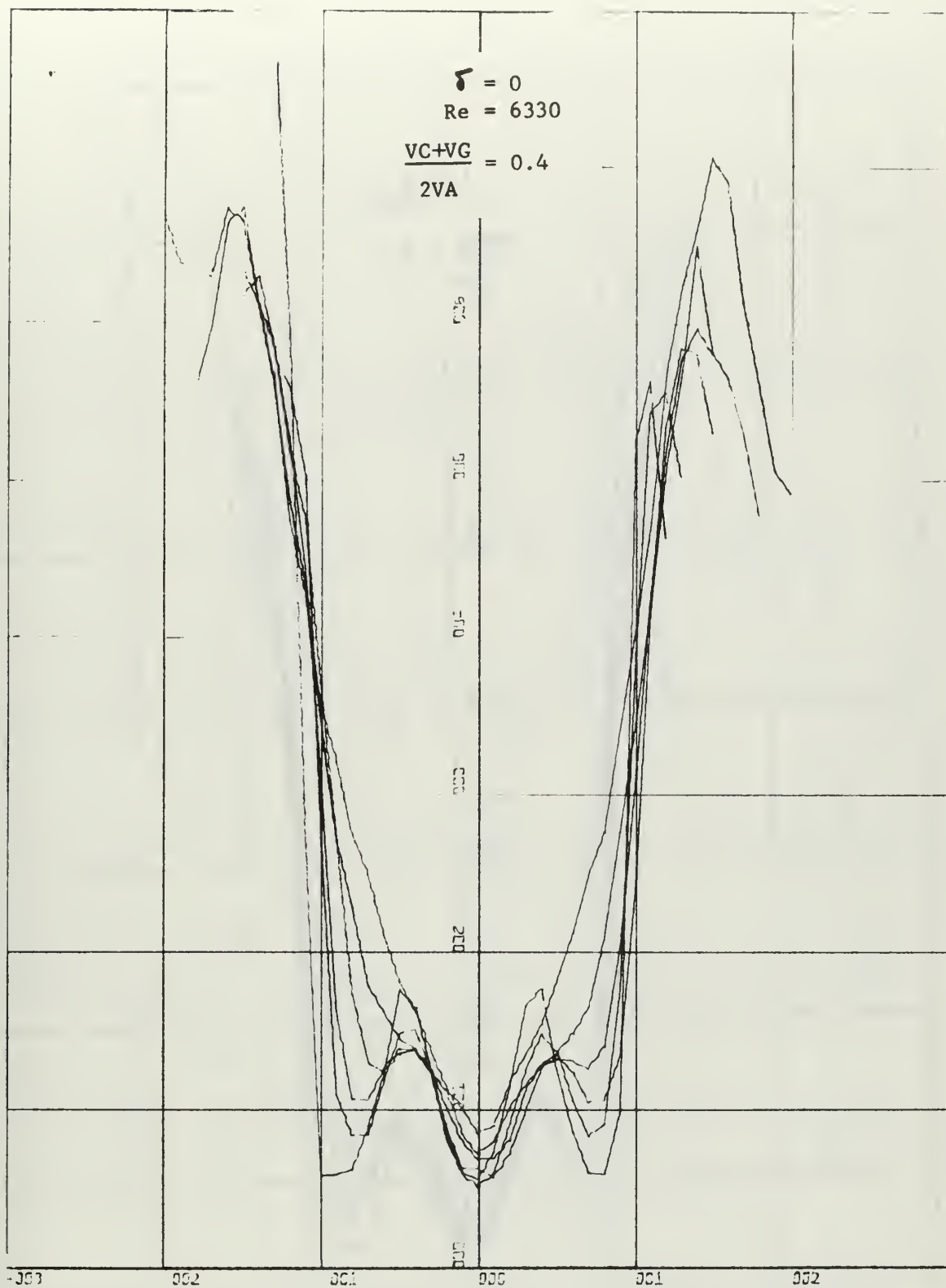


FIGURE 58 TURBULENCE INTENSITY

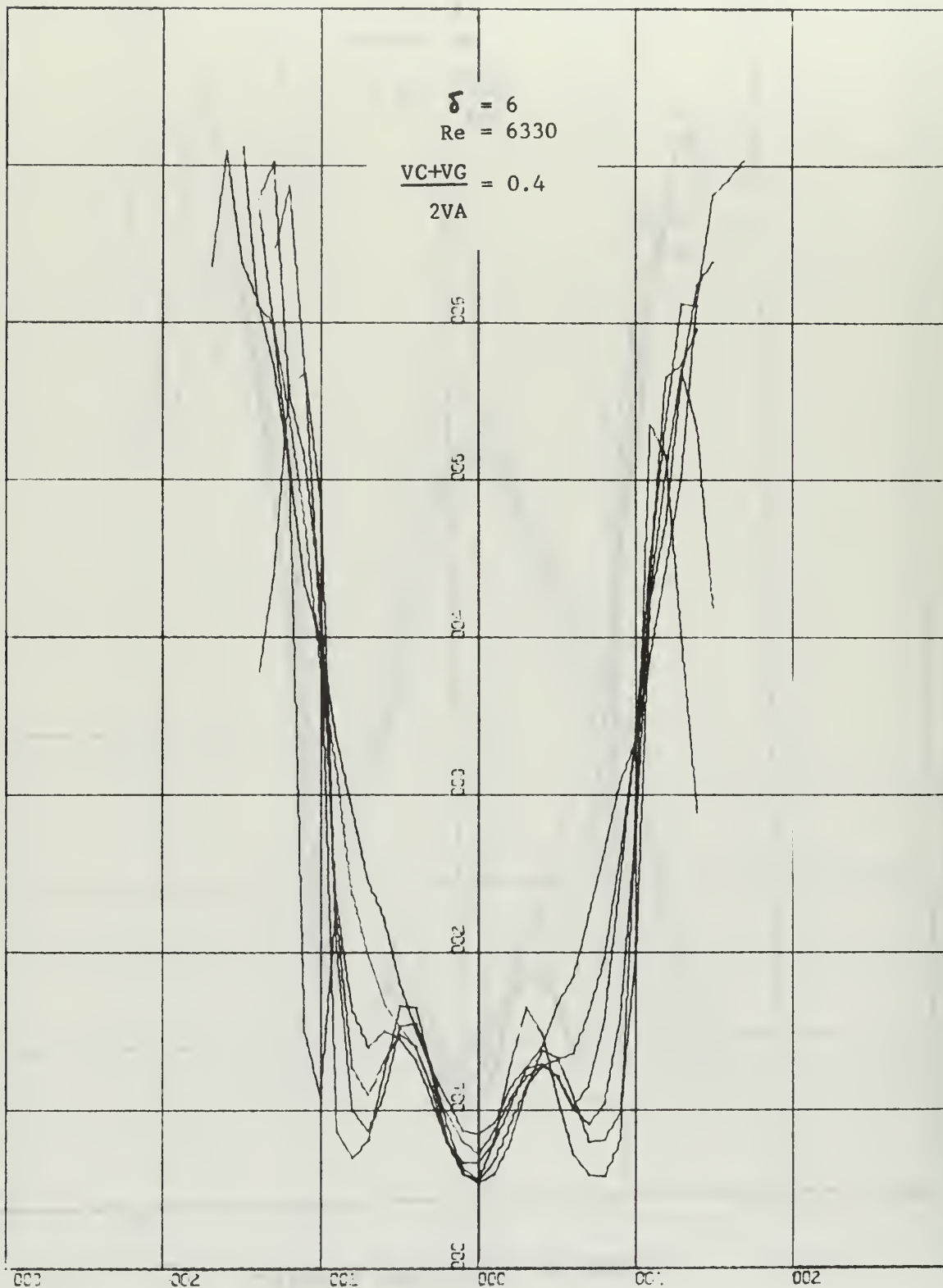


FIGURE 59 TURBULENCE INTENSITY

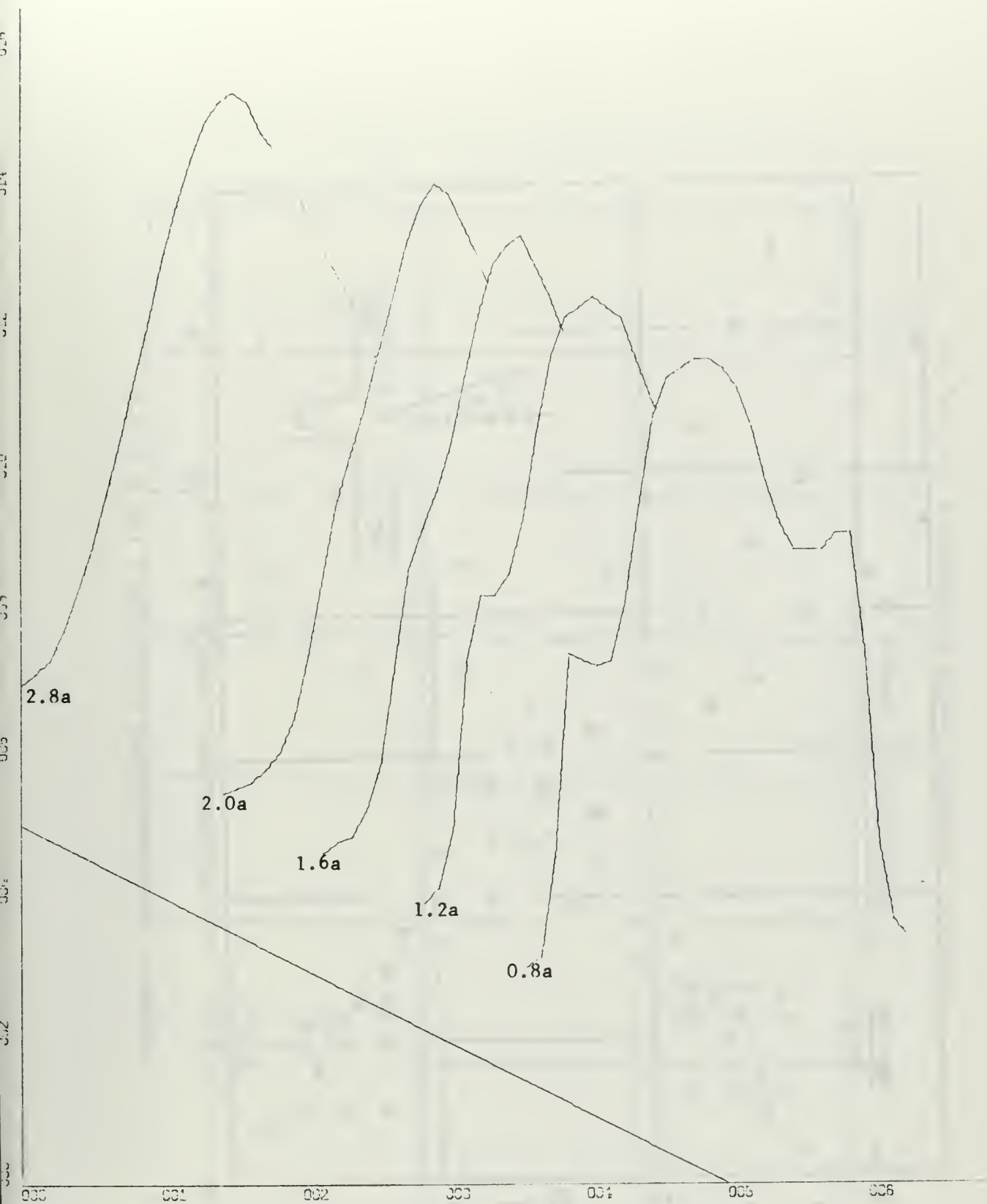


FIGURE 60 VELOCITY PROFILES AT DIFFERENT STATIONS DOWNSTREAM

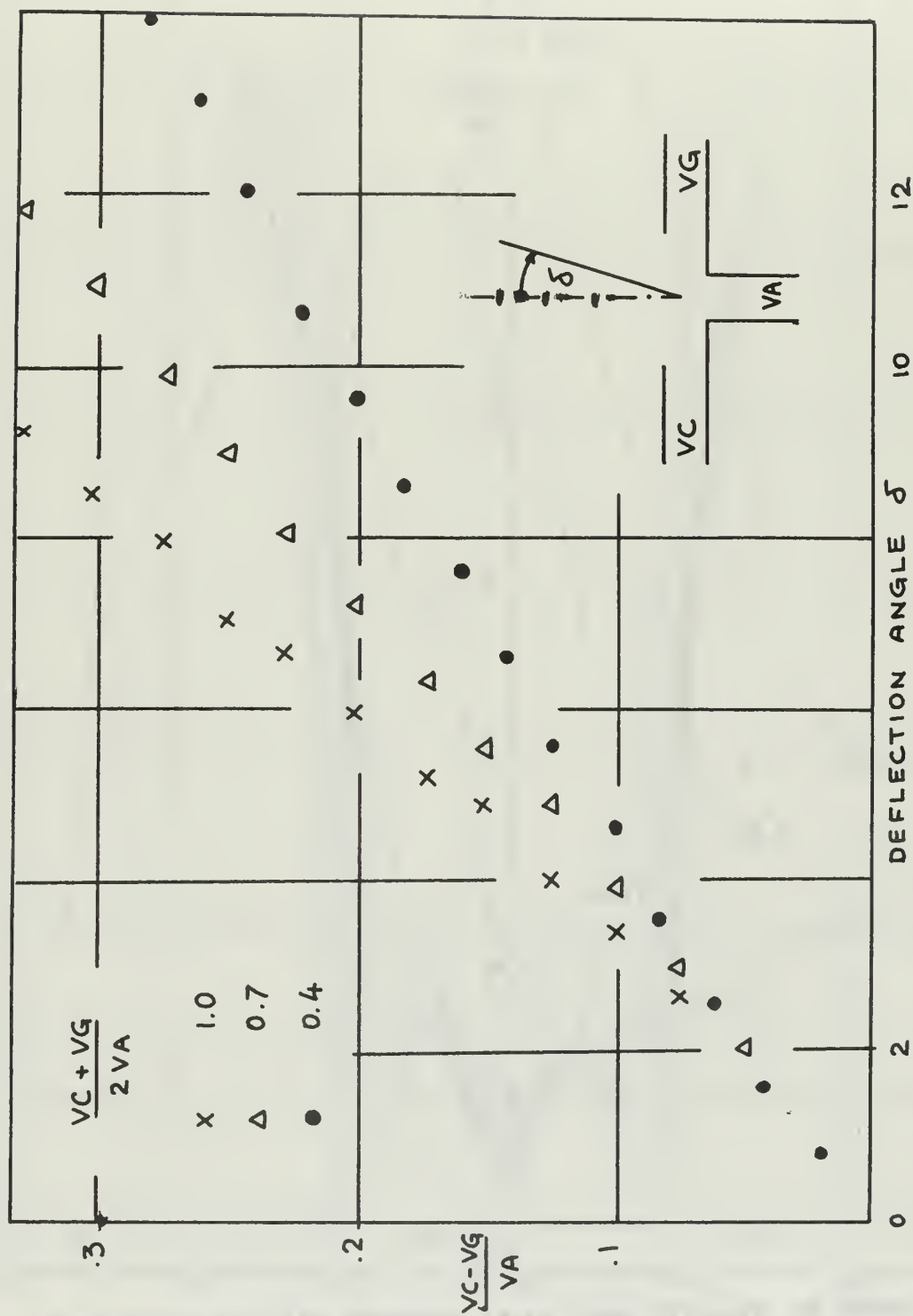


FIGURE 61 DEFLECTION ANGLES $(VC+VG)/2VA = 1.0, 0.7, 0.4$

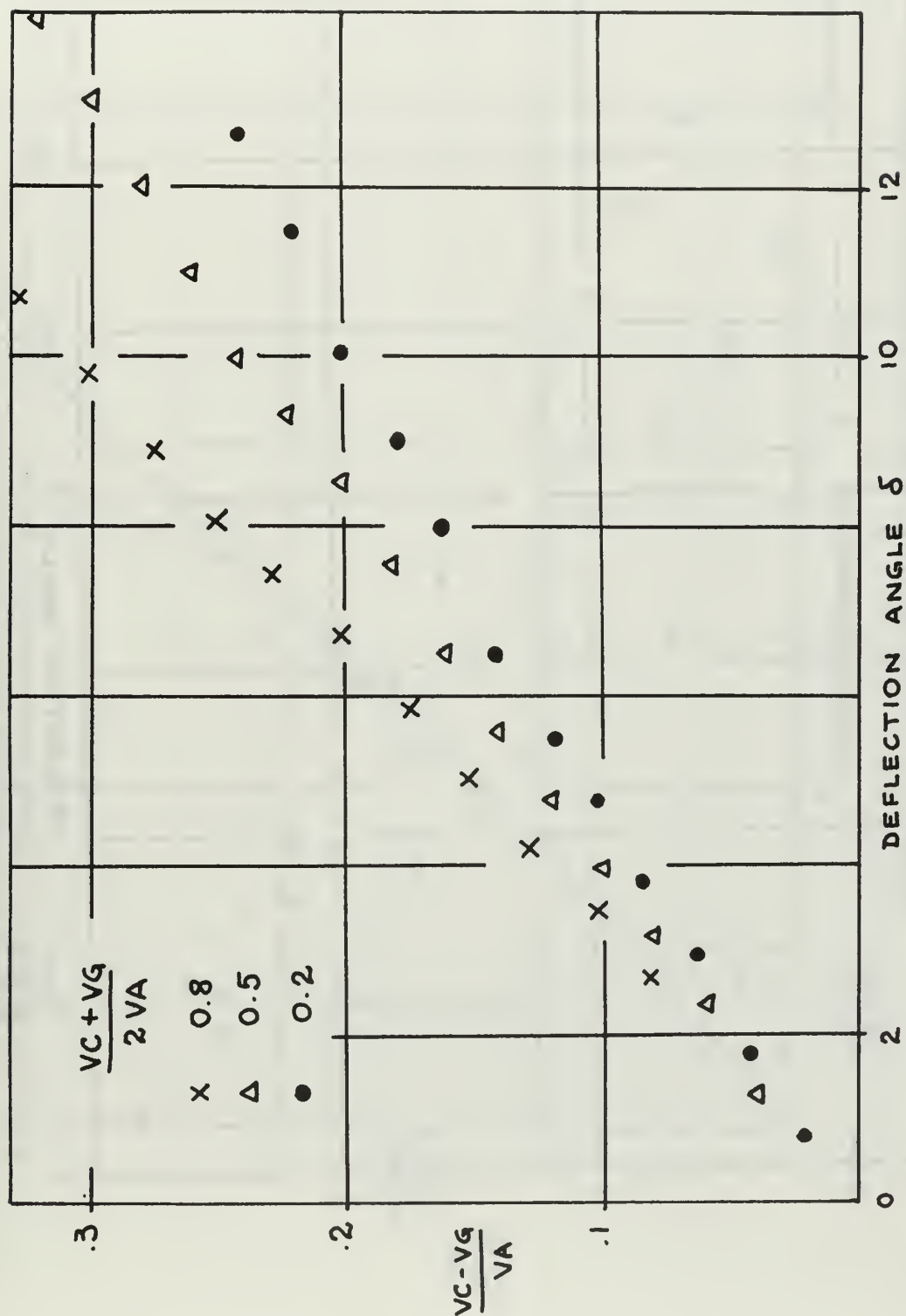


FIGURE 62 DEFLECTION ANGLES $(VC+VG)/2VA = 0.9, 0.6, 0.3$

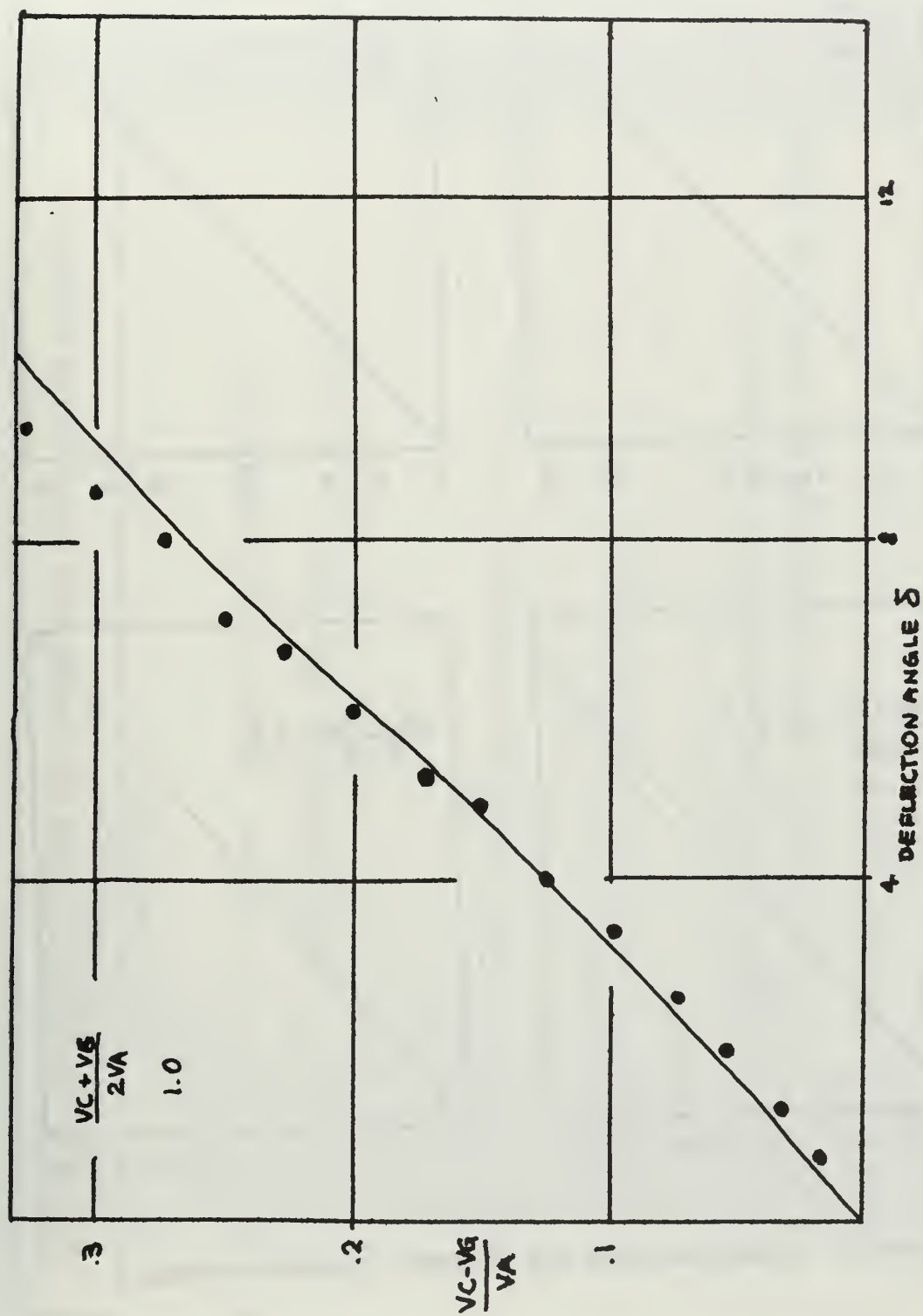


FIGURE 64 EXPERIMENTAL AND THEORETICAL DEFLECTION ANGLES

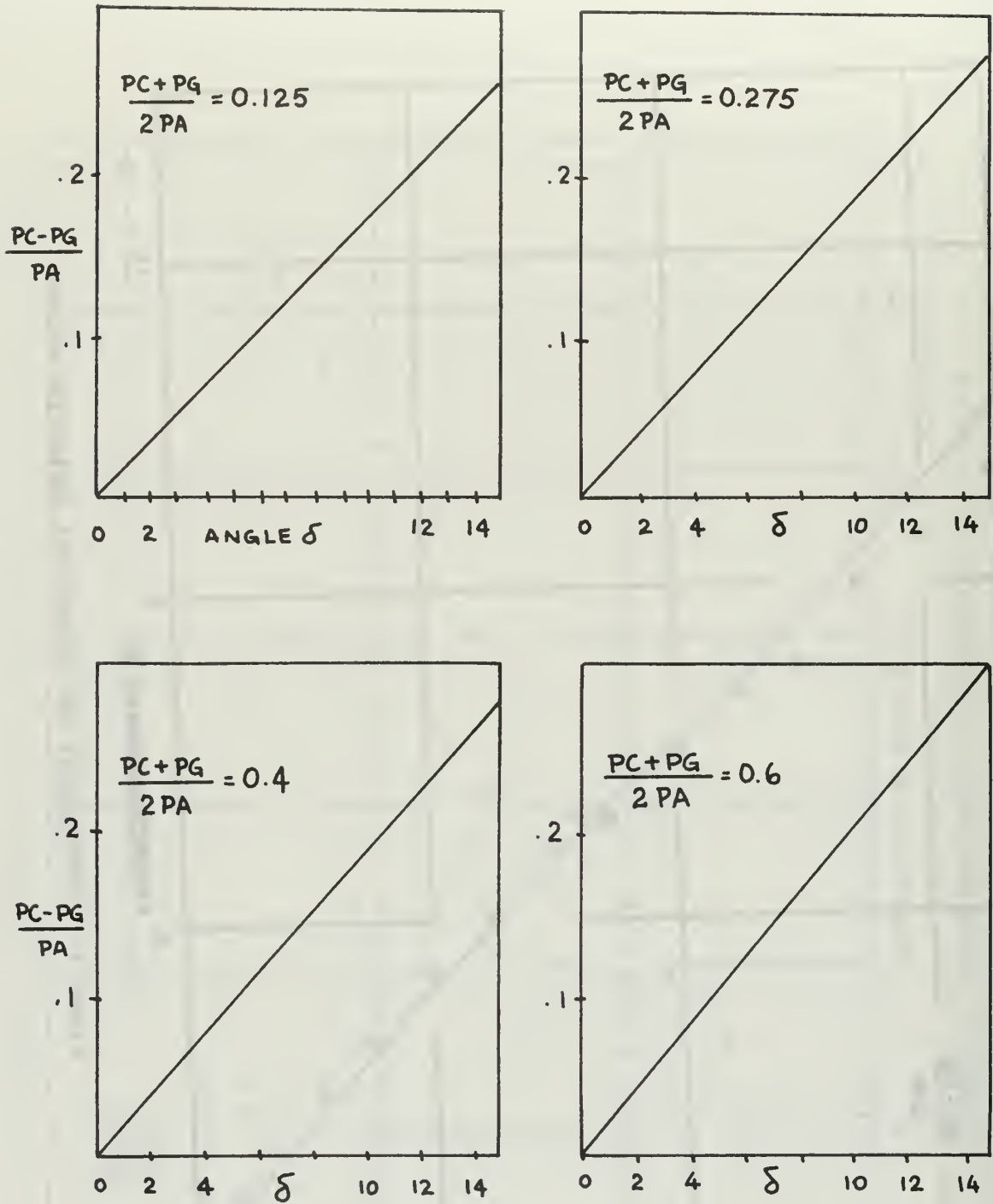


FIGURE 65 DEFLECTION ANGLE FOR DIFFERENT PRESSURES AVERAGE

```

C*****
C PROGRAM A
C ITERATION PROCESS TO SOLVE FOR ANGLE BETA FOR A GIVEN GEOMETRY
C AND VELOCITY RATIOS.
C PROGRAM IN DOUBLE PRECISION
C*****
C IMPLICIT REAL*8(A-H,K,O-Z)
C DEFINE CONSTANTS
  PI=3.1415926D0
  P2=2.0D0/PI
  EPS=0.005D0
C ASSIGN GEOMETRICAL VALUES AOB=A/B , AP2SOB=(A+2S)/B
  AOB=1.0D0
  AP2SOB=3.0D0
C ASSIGN VELOCITY RATIOS VCA=VC/VA , VGA=VG/VA
  VCA=1.1D0
  DO 10 I=1,5
    VCA=VCA-0.1D0
    VGA=VCA
    DO 11 J=1,10
      VCA=VCA+0.02D0
      VGA=VGA-0.02D0
      DVA=VCA-VGA
    C ASSIGN VALUES TO ANGLE BETA FOR ITERATION
      BE=90.0D0
      DO 12 N=1,100
        BE=BE-0.1
        BETA=BE*PI/180.0D0
        S=DSIN(BETA)
    C ASSIGN VALUES TO PARAMETER K FOR ITERATION
      K=0.99D0
      DO 13 M=1,1000
        K=K-C.0002D0
        KK=K*K
      SK=DSQRT(1.0D0-KK)
    C CALCULATE VA (POWER JET VELOCITY)/(RESULTANT JET VELOCITY)
      VA=K/(1.0D0+SK)
    C CALCULATE VC AND VG (VELOCITY RATIO OF CONTROL AND RESULT JET VELOC)
      VC=VCA*VA
      VG=VGA*VA
    C CALCULATE PARAMETERS F,C AND G
      FF=(1.0D0-S*S*KK)/(1.0D0-S*S)
      F=DSQRT(FF)
      SF=DSQRT(FF-KK)
      SSF=DSQRT(FF-1.0D0)
      GG=1.0D0-(1.0D0-KK)*((1.0D0-VG*VG)/(1.0D0+VG*VG))**2
      G=DSQRT(GG)
      SG=DSQRT(GG-KK)

```

```

GER00010
GER00020
GER00030
GER00040
GER00050
GER00060
GER00070
GER00080
GER00090
GER00100
GER00110
GER00120
GER00130
GER00140
GER00150
GER00160
GER00170
GER00180
GER00190
GER00200
GER00210
GER00220
GER00230
GER00240
GER00250
GER00260
GER00270
GER00280
GER00290
GER00300
GER00310
GER00320
GER00330
GER00340
GER00350
GER00360
GER00370
GER00380
GER00390
GER00400
GER00410
GER00420
GER00430
GER00440
GER00450
GER00460
GER00470
GER00480

```

GER00490
 GER00500
 GER00510
 GER00520
 GER00530
 GER00540
 GER00550
 GER00551
 GER00552
 GER00560
 GER00570
 GER00580
 GER00590
 GER00600
 GER00610
 GER00620
 GER00630
 GER00640
 GER00650
 GER00660
 GER00670
 GER00680
 GER00690
 GER00700
 GER00710
 GER00720
 GER00730
 GER00740
 GER00750
 GER00760
 GER00770
 GER00780
 GER00790
 GER00800
 GER00810
 GER00820
 GER00830
 GER00840
 GER00850
 GER00860
 GER00870
 GER00880
 GER00890
 GER00900
 GER00910
 GER00920

GER00930

```

SSG=DSQRT(1.0D0-GG)
CC=1.0D0-((1.0D0-KK)*(1.0D0-VC*VC)/(1.0D0+VC*VC))*2
C=DSQRT(CC)
SSC=DSQRT(CC-KK)
C CONTINUITY EQUATION WHERE DOB=D/B
DOB=VC+VG+VA*AOR
C CORRECTION TO THE CONTRACTION COEFFICIENT
DCB=0.4D0*DOB
C DEFINE TERMS TO BE USED IN EQUATION FOR HORIZONTAL COMPONENT
T1=DOB*S
T2=DARSIN(K)
T3=(VC/SC)*DLOG(SSC/(SC+SK))
T4=(VG/SG)*CLOG(SSG/(SG+SK))
T5=(DOB/SF)*DLOG(SF/(SF+SK))
HORIZ=T1+2.0D0*(SK/PI)*(T2-T3-T4+T5)
C CHECK FOR DIFFERENCE BETWEEN HORIZ AND AP2SOB
IF(DABS(HORIZ-AP2SOB).GT.EPS) GO TO 13
C CALCULATIONS OF THE VERTICAL COMPONENTS WITH ELLIPTIC FUNCTIONS
HH=4.0D0*K/(1.0D0+K)**2
H=DSQRT(HH)
ACC=2.0D0*(K+C)/((K+1.0D0)*(C+1.0D0))
AGG=2.0D0*(K-G)/((1.0D0-G)*(K+1.0D0))
AFF=2.0D0*(K+F)/((K+1.0D0)*(F+1.0D0))
COEF1=PI2*(VC*(1.0D0+C*K)/(C+K)+VG*(1.0D0-G*K)/(K-G)-DOB
  *((1.0D0+F*K)/(F+K))/(K+1.0D0)
COEF2=PI2*(1.0D0-K)/(1.0D0+K)
ZC=VC*(C-1.0D0)/(C+K)
ZG=VG*(G+1.0D0)/(G-K)
ZC=DOB*(F-1.0D0)/(F+K)
CALL DCEL1(EK,H,K)
CALL DCEL3(PIC,ACC,HH)
CALL DCEL3(PIG,AGG,HH)
CALL DCEL3(PIF,AFF,HH)
VERT I=COEF1*EK+COEF2*(ZC*PIC+ZG*PIG-ZD*PIF)-DOB*DCOS(BETA)
C CHECK IF VERTICAL COMPONENT IS ZERO
IF(DABS(VERT).GT.EPS) GO TO 13
WRITE(6,1) PE,DVA,HORIZ,VFRTI
GO TO 11
13 CONTINUE
12 CONTINUE
11 CONTINUE
10 CONTINUE
1 STOP
  FORMAT(4F10.3)
END

```

SUBROUTINE DCEL3(RES,AA,HH)

C SUBROUTINE TO EVALUATE ELLIPTIC INTEGRALS OF THE THIRD KIND

GER00940
GER00950
GER00960
GER00970
GER00980
GER00990
GER01000
GER01010
GER01020
GER01030
GER01040
GER01050
GER01060
GER01070
GER01080
GER01090
GER01100
GER01110
GER01120
GER01130
GER01140
GER01150
GER01160
GER01170
GER01180
GER01190
GER01200
GER01210
GER01220
GER01230

```

      IMPLICIT REAL*8(A-H,O-Z)
      H=DSQRT(HH)
      HPP=1.0D0-HH
      HP=DSQRT(HPP)
      A=DSQRT(DABS(AA))
      SA=DSQRT(DABS(1.0D0-AA))*SA
      CALL DCELI(EK,H,IK)
      CALL DCELI2(FE,H,1.0D0,HPP,IE)
      IF(AA.LT.0.0D0) GO TO 53
      IF(AA.LT.1.0D0.AND.AA.GT.HH) GO TO 52
      IF(AA.LT.HH) GO TO 51
      B=DARSIN(1.0D0/A)
      CALL ZETA(Z,B,HP,EK,EF)
      RES=-A*EK*Z/SSA
      RETURN
51  B=DARSIN(A/H)
      CALL ZETA(Z,B,HP,EK,EE)
      RES=EK+A*EK*Z/SSA
      RETURN
52  B=DARSIN(SA/(A*HP))
      CALL HEUMAN(HEU,B,H,EK,EE)
      RES=3.1415926D0*A*HEU*0.5D0/SSA
      RETURN
53  B=DARSIN(A/SA)
      CALL HEUMAN(HEU,B,H,EK,EE)
      RES=HH*EK/(HH-AA)-3.1415926D0*0.5D0*AA*HEU/(A*SSA)
      RETURN
      END

```

C SUBROUTINE ZETA(Z,ZA,HP,EK,EE)
SUBROUTINE TO EVALUATE ZETA JACOBIAN FUNCTIONS

GER01240
GER01241
GER01250
GER01260
GER01270
GER01280
GER01290
GER01300
GER01310
GER01320

```

      IMPLICIT REAL*8(A-H,O-Z)
      HPP=HP*HP
      ZA=DTAN(ZA)
      CALL DELI2(ZE,ZA,HP,1.0D0,HPP)
      CALL DELI1(ZF,ZA,HP)
      Z=ZE-EE*ZF/EK
      RETURN
      END

```

C SUBROUTINE HEUMAN(HEU,HA,H,EK,EE)
SUBROUTINE TO EVALUATE HEUMAN'S LAMBDA FUNCTIONS

GER01330
GER01331
GER01340
GER01350

```

      IMPLICIT REAL*8(A-H,O-Z)
      HH=H*H

```



```

HA=DTAN(HA)
CALL DELI1(HF,HA,H)
CALL DELI2(HE,HA,H,1.0D0,HH)
HEU=2.0D0*((EE-EK)*(HF+EK*HE))/3.1415926D0
RETURN
END

```

GERO1360
GERO1370
GERO1380
GERO1390
GERO1400
GERO1410

```

C SUBROUTINE CCFL1(RES,AK,IER)
SUBROUTINE TO EVALUATE COMPLETE ELLIPTIC INTEGRALS OF FIRST KIND
IER=0
ARI=2.0
GEO=(0.5-AK)+0.5
GEO=GEO+GEC*AK
RES=0.5
IF(GEO)1,2,4
1 IER=1
2 RES=1.0E75
3 RETURN
4 GEO=GEO*AAARI
GEO=SQRT(GEO)
GEO=GEO+GEO
AAARI=ARI
ARI=ARI+GEO
RES=RES+RES
IF(GEO/AAARI-0.9999)3,5,5
5 RES=RES/ARI*6.283185F0
RETURN
END

```

GERO1420
GERO1421
GERO1430
GERO1440
GERO1450
GERO1460
GERO1470
GERO1480
GERO1490
GERO1500
GERO1510
GERO1520
GERO1530
GERO1540
GERO1550
GERO1560
GERO1570
GERO1580
GERO1590
GERO1600
GERO1610

```

C SUBROUTINE CCFL2(RES,AK,A,B,IER)
SUBROUTINE TO EVALUATE COMPLETE ELLIPTIC INTEGRALS OF SECOND KIND
IER=0
ARI=2.0
GEO=(0.5-AK)+0.5
GEO=GEO+GEC*AK
RES=A
A1=A+B
B0=B+B
IF(GEO)1,2,6
1 IER=1
2 IF(B)3,8,4
3 RES=-1.0E75
4 RETURN
5 RES=1.0E75
6 GEO=GEO*AAARI

```

GERO1620
GERO1621
GERO1630
GERO1640
GERO1650
GERO1660
GERO1670
GERO1680
GERO1690
GERO1700
GERO1710
GERO1720
GERO1730
GERO1740
GERO1750
GERO1760
GERO1770

```

6 GEO=SQRT(GFC)
  GEC=GFO+GEC
  AARI=ARI
  ARI=ARI+GEC
  BO=BO+RES*GEO
  RES=A1
  BO=BO+BO
  A1=BO/ARI+A1
  IF(GEO/AARI-0.9999)5,7,7
7 RES=A1/ARI
  RES=RES+C.57C7963EO*RES
8 RETURN
  END

```

C SUBROUTINE DELI(RES,X,CK)
 SUBROUTINE TO EVALUATE ELLIPTIC INTEGRALS OF THE FIRST KIND

```

1 IF(X)2,1,2
2 RES=0.
3 IF(CK)4,3,4
4 RES=ALOG(ABS(X)+SQRT(1.+X*X))
5 GOTO 13
6 ANGLE=ABS(1./X)
7 GEO=ABS(CK)
8 ARI=1.
9 PIM=0.
10 SQGEO=ARI*GEO
11 ARI=GEO+ARI
12 ANGLE=-SQGEC/ANGLE+ANGLE
13 SQGEO=SQRT(SQGEO)
14 IF(ANGLE)7,6,7 SMALL VALUE
15 REPLACE 0 BY SMALL VALUE

```

C

```

6 ANGLE=SQGEO*1.E-8
7 TEST=AARI*1.E-4
8 IF(ABS(AARI-GEO)-TEST)10,10,8
9 PIM=SQGEO+SQGEO
10 PIM=PIM+PIM
11 IF(ANGLE)9,5,5
12 PIM=PIM+3.1415927
13 GOTO 5
14 IF(ANGLE)11,12,12
15 PIM=PIM+3.1415927
16 RES=(ATAN(ARI/ANGLE)+PIM)/ARI
17 IF(X)14,15,15
18 RES=-RES
19 RETURN

```

GER01730
 GER01790
 GER01800
 GER01810
 GER01820
 GER01830
 GER01840
 GER01850
 GER01860
 GER01870
 GER01880
 GER01890
 GER01900

GER01910
 GER01920
 GER01930
 GER01940
 GER01950
 GER01960
 GER01970
 GER01980
 GER01990
 GER02000
 GER02010
 GER02020
 GER02030
 GER02040
 GER02050
 GER02060
 GER02070
 GER02080
 GER02090
 GER02100
 GER02110
 GER02120
 GER02130
 GER02140
 GER02150
 GER02160
 GER02170
 GER02180
 GER02190
 GER02200
 GER02210
 GER02220
 GER02230


```

C          SUBROUTINE CELI2(R,X,CK,A,B)
C          SUBROUTINE TO EVALUATE ELLIPTIC INTEGRALS OF THE SECOND KIND
C          TEST ARGUMENT
C          IF(X)2,1,2
C          1 R=0.
C          TEST MODULUS
C          2 C=0.
C          D=0.5
C          IF(CK)7,3,7
C          3 R=SQRT(1.+X*X)
C          R=(A-B)*ABS(X)/R+B*A*LOG(ABS(X)+R)
C          TEST SIGN OF ARGUMENT
C          4 R=R+C*(A-B)
C          IF(X)5,6,6
C          5 R=-R
C          6 RETURN
C          INITIALIZATION
C          7 AN=(B+A)*0.5
C          AA=A
C          R=B
C          ANG=ABS(1./X)
C          PIM=0.
C          ISI=0
C          ARI=1.
C          GEO=ABS(CK)
C          LANDEN TRANSFORMATION
C          8 R=AA*GEO+R
C          SGEO=ARI*GEO
C          AA=AN
C          AARI=ARI
C          ARITHMETIC MEAN
C          ARI=GEO+ARI
C          SUM OF SINE VALUES
C          AN=(R/ARI+AA)*0.5
C          AANG=ABS(ANG)
C          ANG=-SGEO/ANG+ANG
C          PIMA=PIM
C          IF(ANG)10,9,11
C          9 ANG=-1.E-8*AANG
C          PIM=PIM+3.1415927
C          10 ISI=ISI+1
C          11 AANG=ARI*ARI+ANG*ANG
C          P=D/SQRT(AANG)
C          IF(ISI-4)13,12,12

```

```

GER02240
GER02250
GER02251
GER02260
GER02270
GER02280
GER02290
GER02300
GER02310
GER02320
GER02330
GER02340
GER02350
GER02360
GER02370
GER02380
GER02390
GER02400
GER02410
GER02420
GER02430
GER02440
GER02450
GER02460
GER02470
GER02480
GER02490
GER02500
GER02510
GER02520
GER02530
GER02540
GER02550
GER02560
GER02570
GER02580
GER02590
GER02600
GER02610
GER02620
GER02630
GER02640
GER02650
GER02660
GER02670
GER02680

```

```

12 ISI=ISI-4
13 IF(ISI-2)15,14,14
14 P=-P
15 C=C+P
    D=D*(AARI-GEO)*0.5/ARI
    IF(ABS(AARI-GEO)-1.E-4*AARI)17,17,16
16 SGEO=SQRT(SGEO)
    C
    GEOMETRIC MEAN
    GEO=SGEO+SGEO
    PIM=PIM+PIMA
    ISI=ISI+ISI
    GOTQ 8
    ACCURACY WAS SUFFICIENT
C 17 R=(ATAN(ARI/ANG)+PIM)*AN/ARI
    C=C+D*ANG/AANG
    GOTQ 4
    END

```

```

GER02690
GER02700
GER02710
GER02720
GER02730
GER02740
GER02750
GER02760
GER02770
GER02780
GER02790
GER02800
GER02810
GER02820
GER02830
GER02840
GER02850

```


[illegible]

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KEY WORDS

LINK A

LINK B

LINK C

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